

A c_0 SATURATED BANACH SPACE WITH TIGHT STRUCTURE

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Dedicated to the memory of Nigel J. Kalton

ABSTRACT. It is shown that variants of the HI methods could yield objects closely connected to the classical Banach spaces. Thus we present a new c_0 saturated space, denoted as \mathfrak{X}_0 , with rather tight structure. The space \mathfrak{X}_0 is not embedded into a space with an unconditional basis and its complemented subspaces have the following structure. Everyone is either of type I, namely, contains an isomorph of \mathfrak{X}_0 itself or else is isomorphic to a subspace of c_0 (type II). Furthermore for any analytic decomposition of \mathfrak{X}_0 into two subspaces one is of type I and the other is of type II. The operators of \mathfrak{X}_0 share common features with those of HI spaces.

Introduction

The aim of the present paper is to provide a new norm on $c_{00}(\mathbb{N})$ resulting a c_0 saturated Banach space. This norm is defined with the use of a modification of the standard method yielding Hereditarily Indecomposable (HI) Banach spaces. This approach reveals a Banach space which is c_0 saturated but also has a rather tight structure. The following describes the main properties of the space.

Theorem A There exists a separable Banach space \mathfrak{X}_0 satisfying the following properties.

- (i) The space \mathfrak{X}_0 is c_0 saturated and it is not embedded into a space with an unconditional basis.
- (ii) The dual space \mathfrak{X}_0^* is separable.
- (iii) Every complemented subspace Y of \mathfrak{X}_0 is of one of the following two types. Either \mathfrak{X}_0 is isomorphic to a subspace of Y (type I) or Y is isomorphic to a subspace of c_0 (type II).
- (iv) If $\mathfrak{X}_0 = Y \oplus Z$ with Y, Z of infinite dimension, then one of Y, Z is of type I and the other is of type II. Moreover if the type II complemented subspace is isomorphic to c_0 , then the other one is isomorphic to \mathfrak{X}_0 . In particular \mathfrak{X}_0 is not isomorphic to its square $\mathfrak{X}_0 \oplus \mathfrak{X}_0$.

Note that properties (iii) and (iv) reminds the strictly quasi-prime spaces introduced in [AR]. It is open if the space \mathfrak{X}_0 is strictly quasi-prime. The difference between \mathfrak{X}_0 and the examples of strictly-quasi prime spaces presented in [AR] is that the later spaces are not c_0 or ℓ^p saturated.

The definition of the norm of \mathfrak{X}_0 goes as follows. We fix two appropriate increasing sequences $(m_j)_{j \in \mathbb{N}}, (n_j)_{j \in \mathbb{N}}$ of natural numbers. In the first stage we define a norming set G_0 as follows. The set G_0 is the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following properties:

- (1) It contains the natural basis $(e_n)_{n \in \mathbb{N}}$ of $c_{00}(\mathbb{N})$ and it is symmetric.
- (2) It is closed under the even operations $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operation for every $j \in \mathbb{N}$.

We recall that this means that for every $j \in \mathbb{N}$, $d \leq n_{2j}$ and $f_1 < \dots < f_d$ in G_0 the functional $\frac{1}{m_{2j}} \sum_{i=1}^d f_i$ belongs to G_0 . Also, as usual, we set $w(f) = m_{2j}$ (the weight of f) if f is a result of a $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operation.

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- (3) G_0 contains all $f = \sum_{i=1}^n a_i f_i$, where $\sum_{i=1}^n a_i^2 \leq 1$ and $\{f_i\}_{i=1}^n$ elements of G_0 with pairwise different weights.

The norm induced by G_0 on $c_{00}(\mathbb{N})$ is denoted as $\|\cdot\|_{G_0}$. Finally we set \mathfrak{X}_{G_0} the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{G_0})$. The space \mathfrak{X}_{G_0} , which is a reflexive one with an unconditional basis, is a variant of E. Odell and Th. Schlumprecht space (cf [OS1]) having no ℓ^p as a spreading model. We refer the interested reader to [AMP] or [AKT] for a further study of spaces with similar properties.

Next we extend the set G_0 to W_0 which yields the norm of the space \mathfrak{X}_0 as follows. First we consider a coding function σ similar to the one used in the definitions of HI and related spaces. Using that coding we define the $\sigma - n_{2j+1}$ special sequences $(f_i)_{i=1}^{n_{2j+1}}$, where each f_i belongs to G_0 . Finally we set

$$W_0 = G_0 \cup \left\{ E \left(\frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i \right) : E \text{ interval of } \mathbb{N} \text{ and } (f_i)_{i=1}^{n_{2j+1}} \text{ is a } \sigma - n_{2j+1} \text{ special sequence} \right\}.$$

The space \mathfrak{X}_0 is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{W_0})$. Let us point out that the main difference of \mathfrak{X}_0 from a standard HI example (for example the Gowers-Maurey space (cf [GM])) is that here we use the odd (i.e. conditional) operations only once at the final step of the definition of W_0 . The familiar reader will also observe that the usual definition of a HI space does not use condition (3) of the definition of G_0 . Condition (3) is critical for proving many of the properties of \mathfrak{X}_0 as well as $\mathfrak{L}(\mathfrak{X}_0)$. The particular use of this, is in Proposition 2.1. However it is worth noticing that the variant of the definition not including condition (3) of G_0 also yields a c_0 saturated space and we do not know if this space satisfies the further properties of the space \mathfrak{X}_0 .

As $G_0 \subset W_0$ the identity operator $id : \mathfrak{X}_0 \longrightarrow \mathfrak{X}_{G_0}$ is clearly continuous. The understanding of the behavior of id on the subspaces of \mathfrak{X}_0 is essential for studying the structure of the space. In this direction we have the following.

Proposition B

- (1) The operator $id : \mathfrak{X}_0 \longrightarrow \mathfrak{X}_{G_0}$ is strictly singular.
- (2) Let $(x_n)_{n \in \mathbb{N}}$ be a normalized sequence in \mathfrak{X}_0 such that $\lim_{n \rightarrow \infty} \|x_n\|_{G_0} = 0$. Then there exists a subsequence $(x_n)_{n \in L}$ which is equivalent to the c_0 basis.
- (3) Let Y be a subspace of \mathfrak{X}_0 such that $id|_Y : Y \longrightarrow \mathfrak{X}_{G_0}$ is compact. Then Y is isomorphic to a subspace of c_0 .
- (4) Let Y, Z be infinite dimensional subspaces of \mathfrak{X}_0 such that $id|_Y, id|_Z$ are not compact operators. Then $d(S_Y, S_Z) = 0$.

Properties (1) and (2) of Proposition B yield that \mathfrak{X}_0 is indeed c_0 saturated. The proof of property (3) requires some beautiful and advanced concepts and results due to Kalton (cf [K]). Indeed we actually show that any subspace Y such that $id|_Y$ is a compact operator satisfies the c_0 tree property (see Definition 5.2), which according to Kalton (cf [K], Thm. 3.2) yields that Y is isomorphic to a subspace of c_0 . A consequence of properties (3) and (4) is that every complemented subspace of \mathfrak{X}_0 is either of type I or of type II (see Thm. A).

A second result describing the tight structure of \mathfrak{X}_0 and its relation to HI spaces concerns the operators. Let point out that Sobczyk's theorem (cf [SO]) yields that \mathfrak{X}_0 admits many projections as every c_0 subspace is a complemented one. The following explains that the non strictly singular operators on \mathfrak{X}_0 have a precise structure.

Theorem C Every bounded linear operator $T : \mathfrak{X}_0 \longrightarrow \mathfrak{X}_0$ is of the form $T = \lambda \cdot I + S$ with S satisfying the following. If there exists a subspace Y of \mathfrak{X}_0 with $S|_Y$ is an isomorphism, then Y is isomorphic to a subspace of c_0 .

The paper is organized into six sections. In the first one we present the definition of the norming sets G_0 and W_0 and the corresponding spaces \mathfrak{X}_{G_0} and \mathfrak{X}_0 . The second one is devoted to the study of bounded block sequences $(x_n)_{n \in \mathbb{N}}$ in \mathfrak{X}_0 satisfying $\lim_{n \rightarrow \infty} \|x_n\|_{G_0} > 0$. The main result of this section (Prop. 2.2) asserts that any such sequence contains arbitrarily large seminormalized averages on

which the G_0 and W_0 norms coincide. Sections 3 and 4 are devoted to the basic inequality, the exact pairs and the dependent sequences. All these are closely related to the corresponding concepts and results appeared and used in the study of HI spaces (cf [AMP]). Sections 5 and 6 include the proofs of Theorems A and C respectively.

We make use of the following standard notation throughout this article.

- i. We denote by $c_{00}(\mathbb{N})$ the vector space $c_{00}(\mathbb{N}) = \{f : \mathbb{N} \rightarrow \mathbb{R} : f(n) \neq 0 \text{ for finitely many } n \in \mathbb{N}\}$ and by $c_{00}^{\mathbb{Q}}(\mathbb{N})$ the set of all elements of $c_{00}(\mathbb{N})$ with rational coordinates. For every $x \in c_{00}(\mathbb{N})$ we denote by $\text{supp } x$ the set $\text{supp } x = \{n \in \mathbb{N} : x(n) \neq 0\}$ and by $\text{ran } x$ the minimal interval of \mathbb{N} that contains $\text{supp } x$.
- ii. We denote by $(e_n)_n$ the standard Hamel basis of $c_{00}(\mathbb{N})$, which will also be considered as functionals on $c_{00}(\mathbb{N})$ acting through the usual inner product and denoted as $(e_n^*)_n$.
- iii. Let E_1, E_2 be two nonempty finite subsets of \mathbb{N} . We write $E_1 < E_2$ if $\max E_1 < \min E_2$. Also for a $n \in \mathbb{N}$, we write $n < E_1$ if $\{n\} < E_1$. If x_1, x_2 are non zero sequences of $c_{00}(\mathbb{N})$ we write $x_1 < x_2$ whenever $\text{ran } x_1 < \text{ran } x_2$. In addition for a sequence $f : \mathbb{N} \rightarrow \mathbb{R}$ and E an interval of \mathbb{N} we denote by Ef the sequence $f \cdot X_E$, where X_E is the characteristic function of E .
- iv. We say that a subset F of $c_{00}(\mathbb{N})$ is closed under the (\mathcal{A}_n, θ) -operation for $n \in \mathbb{N}$ and $0 < \theta < 1$, if for every $d \leq n$ and for every $f_1 < \dots < f_d$ in F we have that $\theta \sum_{i=1}^d f_i \in F$.
- v. We say that a subset F of $c_{00}(\mathbb{N})$ is symmetric, if for every $f \in F$ it follows that $-f \in F$.
- vi. We say that a subset F of $c_{00}(\mathbb{N})$ is closed in restrictions to finite intervals of \mathbb{N} , if for every $f \in F$ and E finite interval of \mathbb{N} , it follows that $Ef \in F$.
- vii. Let L an infinite subset of \mathbb{N} and $k \in \mathbb{N}$. We denote by $[L]^k$ the set of all subsets of \mathbb{N} with k elements and with $[L]$ the set of all infinite subsets of L .

1. The norming set of the Banach space \mathfrak{X}_0

In this section we define the norming sets G_0 and W_0 yielding the spaces \mathfrak{X}_{G_0} and \mathfrak{X}_0 respectively. We fix two sequences of natural numbers $(m_j)_j$ and $(n_j)_j$ defined recursively as follows. We set $m_1 = 2^8$ and $m_{j+1} = m_j^5$ and $n_1 = 2^7$ and $n_{j+1} = (2n_j)^{s_{j+1}}$ where $s_{j+1} = \log_2(m_{j+1}^4)$, $j \geq 1$.

Definition 1.1. *Let G_0 be the minimal subset of $c_{00}(\mathbb{N})$ satisfying the following:*

- (1) G_0 contains the set $F_0 = \{e_n^* : n \in \mathbb{N}\}$.
- (2) G_0 is symmetric.
- (3) G_0 is closed under the $(\mathcal{A}_{n_{2j}}, \frac{1}{m_{2j}})$ operation for every $j \in \mathbb{N}$.
- (4) It contains the set $\{\sum_{i=1}^d a_i f_i : d \in \mathbb{N}, a_i \in \mathbb{Q}, \sum_{i=1}^d a_i^2 \leq 1 \text{ and } f_i \in G_0 \text{ with } (w(f_i))_{i=1}^d \text{ pairwise different}\}$.

For an $f \in G_0$ we say that f has weight m_{2j} and we write $w(f) = m_{2j}$ if and only if there exists $d \in \mathbb{N}$ with $d \leq n_{2j}$ and $f_1 < \dots < f_d$ in G_0 such that $f = \frac{1}{m_{2j}} \sum_{i=1}^d f_i$. Such an f is called a functional with weight.

Definition 1.2. *We define the Banach space \mathfrak{X}_{G_0} to be the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{G_0})$, where*

$$\|x\|_{G_0} = \sup\{|f(x)| : f \in G_0\}, x \in c_{00}(\mathbb{N}).$$

The space \mathfrak{X}_{G_0} resembles the space defined by E. Odell and Th. Schlumprecht in [OS1], where it is proved that this space does not admit c_0 and ℓ^p as a spreading model. Since the space \mathfrak{X}_{G_0} does not admit c_0 and ℓ^p as a spreading model for every $1 \leq p < \infty$ (for a proof we refer to [OS1] or [AMP]) and the basis is unconditional, it follows that this space is reflexive. Next we define the $\sigma - n_{2j+1}$ special sequences and the norming set W_0 .

Definition 1.3. Let Ω_1, Ω_2 be two disjoint infinite subsets of \mathbb{N} and $Q_s = \{(f_1, \dots, f_d) : d \in \mathbb{N}, f_i \in G_0, f_i \neq 0, i = 1, \dots, d, f_1 < \dots < f_d\}$. Since Q_s is a subset of $c_{00}^{\mathbb{Q}}(\mathbb{N})$, it follows that Q_s is countable, so we may select an injective map $\sigma : Q_s \rightarrow \{2j : j \in \Omega_2\}$ such that

$$m_{\sigma(f_1, \dots, f_d)} > \max\left\{\frac{1}{|f_i(e_l)|} : i = 1, \dots, d, l \in \text{supp}(f_i)\right\} \cdot \max \text{supp}(f_d)$$

for every (f_1, \dots, f_d) in Q_s .

Let $j \in \mathbb{N}$. A finite sequence $(f_i)_{i=1}^{n_{2j+1}}$ with $(f_1, \dots, f_{n_{2j+1}})$ in Q_s , is said to be $\sigma - n_{2j+1}$ special sequence provided:

- (1) each f_i is a functional with weight.
- (2) $w(f_1) = m_{2j_1}$, $j_1 \in \Omega_1$ and $n_{2j+1}^2 < m_{2j_1}$.
- (3) $w(f_{i+1}) = m_{\sigma(f_1, \dots, f_i)}$, for all $i \in \{1, \dots, n_{2j+1} - 1\}$.

We pass now to define the norming set W_0 and the corresponding space \mathfrak{X}_0 .

Definition 1.4. Let W_0 be the minimal subset of $c_{00}(\mathbb{N})$ such that

- (1) $G_0 \cup \left\{\frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i : f_i \in G_0 \text{ with } (f_i)_{i=1}^{n_{2j+1}} \sigma - n_{2j+1} \text{ special sequence}\right\} \subset W_0$
- (2) W_0 is symmetric
- (3) W_0 is closed in restrictions to the finite intervals of \mathbb{N} .

The Banach space \mathfrak{X}_0 is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{W_0})$, where

$$\|x\|_{W_0} = \sup\{|f(x)| : f \in W_0\}, x \in c_{00}(\mathbb{N}).$$

For an $f \in W_0$ we say that f has weight m_{2j+1} and we write $w(f) = m_{2j+1}$ if and only if there exists a $\sigma - n_{2j+1}$ special sequence $(f_i)_{i=1}^{n_{2j+1}}$ in G_0 such that $f = \epsilon E \frac{1}{m_{2j+1}} \sum_{i=1}^d f_i$, where $|\epsilon| = 1$ and E a finite interval of \mathbb{N} .

Remark 1.1. (1) The norming set G_0 is closed in restrictions to the finite subsets of \mathbb{N} .

- (2) It is easily checked that $W_0 = G_0 \cup \left\{\epsilon E \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i : |\epsilon| = 1, E \text{ finite interval of } \mathbb{N} \text{ and } f_i \in G_0 \text{ with } (f_i)_{i=1}^{n_{2j+1}} \sigma - n_{2j+1} \text{ special sequence}\right\}$.
- (3) If $f \in W_0$, then $\|f\|_{\infty} \leq 1$.
- (4) The basis $(e_n)_{n \in \mathbb{N}}$ of the Banach space \mathfrak{X}_0 is bimonotone and $\|e_n\|_{W_0} = 1$ for all $n \in \mathbb{N}$. Also the basis of \mathfrak{X}_{G_0} is 1-unconditional (i.e. for every $x \in \mathfrak{X}_{G_0}$ and E subset of \mathbb{N} we have that $\|Ex\|_{G_0} \leq \|x\|_{G_0}$).

2. Estimating averages in \mathfrak{X}_0

The main result of this section is the following proposition, which is a key ingredient for studying the structure of the space \mathfrak{X}_0 and the corresponding one of $\mathfrak{L}(\mathfrak{X}_0)$.

Proposition 4.1. For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ with $k > n$ and every block sequence $(x_n)_{n \in \mathbb{N}}$ with

$$0 < \epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

there exists an $L \in [\mathbb{N}]$ such that

for every $n_1 < \dots < n_k$ in L and every $\phi \in (W_0 \setminus G_0)$ it follows that

$$\phi\left(\frac{x_{n_1} + \dots + x_{n_k}}{k}\right) \leq \left\|\frac{x_{n_1} + \dots + x_{n_k}}{k}\right\|_{G_0}.$$

Hence

$$\left\|\frac{x_{n_1} + \dots + x_{n_k}}{k}\right\|_{W_0} = \left\|\frac{x_{n_1} + \dots + x_{n_k}}{k}\right\|_{G_0}.$$

The proof of the proposition is of combinatorial nature, i.e. is mainly based on Ramsey's theorem [R], and uses the property that the norming set G_0 is closed under rational convex combinations. Actually this is the only point where that property is used. A consequence of the proposition is that every block sequence $(x_n)_n$ with $\overline{\lim}\|x_n\|_{W_0} < \infty$ and $\underline{\lim}\|x_n\|_{G_0} > 0$ admits further block sequences which are arbitrarily large seminormalized l^1 averages and moreover their norm in \mathfrak{X}_0 coincide with the corresponding one in \mathfrak{X}_{G_0} . This is the fundamental ingredient for proving the properties of the space \mathfrak{X}_0 and the properties of $\mathfrak{L}(\mathfrak{X}_0)$, as it permits to pass to exact pairs in \mathfrak{X}_0 and then to dependent sequences in \mathfrak{X}_0 .

Definition 2.1. Let $k \in \mathbb{N}$, $\delta > 0$ and $(x_n)_{n=1}^k$ be a finite block sequence. Let also ϕ be a functional in $(W_0 \setminus G_0)$ of the form $\phi = \frac{1}{m_{2j+1}} \sum_{i=1}^q f_i$, where $q \leq n_{2j+1}$ and $f_1 < \dots < f_q$ in G_0 .

- (1) We will say that $(x_n)_{n=1}^k$ is (ϕ, δ) separated if there exist
 - i. $d_1 < d_2 < d_3$ in $\{1, \dots, k\}$ and
 - ii. $E_{d_1} < E_{d_2} < E_{d_3}$ subintervals of $\{1, \dots, q\}$
such that $\delta < \sum_{p \in E_{d_i}} f_p(x_{d_i})$ for all $i = 1, 2, 3$.
- (2) We will say that $(x_n)_{n=1}^k$ is δ -separated if there exists a functional ϕ in $(W_0 \setminus G_0)$ such that $(x_n)_{n=1}^k$ is (ϕ, δ) separated.

Notation 2.1. Let $k \in \mathbb{N}$, $n_1 < \dots < n_k$ in \mathbb{N} and $(x_{n_i})_{i=1}^k$ be a finite block sequence. Let also $\phi \in (W_0 \setminus G_0)$ with $\phi = \frac{1}{m_{2j+1}} \sum_{i=1}^q f_i$, where $q \leq n_{2j+1}$ and $(f_i)_{i=1}^q$ successive elements in G_0 .

Then for every $d \in \{1, \dots, k\}$ we define the set $E_{n_d}^\phi$ as follows:

$$E_{n_d}^\phi = \{i \in \{1, \dots, q\} : \text{ran}(f_i) \cap \text{ran}(x_{n_d}) \neq \emptyset\}.$$

Lemma 2.1. Let $k \in \mathbb{N}$, $\delta > 0$ and $(x_n)_{n \in \mathbb{N}}$ be a block sequence. Let also $L \in [\mathbb{N}]$ satisfying that for every $n_1 < \dots < n_k$ in L there exists $\phi \in (W_0 \setminus G_0)$ such that the block sequence $(x_{n_i})_{i=1}^k$ is (ϕ, δ) separated. Then for every $M > 0$, there exists $L_0 \in [L]$ such that $M \leq \|x_n\|_{G_0}$ for all $n \in L_0$.

Proof: Assume on the contrary. Then there exist $M > 0$ and $P \in [L]$ (assume without loss of generality that $P = L$) such that $\|x_n\|_{G_0} \leq M$ for all $n \in L$.

From the fact that for every $n_1 < \dots < n_k$ in L there exists $\phi \in (W_0 \setminus G_0)$ such that the block sequence $(x_{n_i})_{i=1}^k$ is (ϕ, δ) separated, applying Ramsey's Theorem [R] for the set $[L]^k$, (i.e. if $[L]^k = A \cup B$, then there exists $L_1 \in [L]$ such that $[L_1]^k \subset A$ or $[L_1]^k \subset B$), we conclude that

- (1) there exists $L_1 \in [L]$ and
- (2) there exist $d_1 < d_2 < d_3$ in $\{1, \dots, k\}$

such that for every $F = \{n_1 < \dots < n_k\} \in [L_1]^k$, there exists $\phi_F \in (W_0 \setminus G_0)$ of the form

$\phi_F = \frac{1}{m_{2j_F+1}} \sum_{i=1}^{d_F} f_i^F$, where $d_F \leq n_{2j_F+1}$, $f_1^F < \dots < f_{d_F}^F$ in G_0 and $E_{d_1}^F < E_{d_2}^F < E_{d_3}^F$ subintervals of $\{1, \dots, d_F\}$ such that

$$(2.1) \quad \delta < \sum_{q \in E_{d_i}^F} f_q^F(x_{n_{d_i}}) \text{ for all } i = 1, 2, 3.$$

For every $F \in [L_1]^k$ we fix the functional ϕ_F . Throughout this proof, the functional ϕ_F will be called the corresponding functional to F .

Let $L_1 = \{l_1 < \dots < l_{d_1} < \dots\}$ and $L_{1,d_1} = L_1 \setminus \{l_1 < \dots < l_{d_1}\}$.

Moreover if $F \in [L_{1,d_1}]^{k-d_1}$, we denote by \overline{F} the set $\overline{F} = \{l_1 < \dots < l_{d_1}\} \cup F$.

Claim: There exists $C > 0$ such that for every $F \in [L_{1,d_1}]^{k-d_1}$ we have that $w(\phi_{\overline{F}}) \leq C$.

Assume not. Then for $C = \frac{|\text{supp}(x_{l_{d_1}})|}{\delta}$, there exists $F \in [L_{1,d_1}]^{k-d_1}$ such that if $\phi_{\overline{F}}$ is the

corresponding functional to \overline{F} , then from (2.1) we get that

$$\delta < \sum_{q \in E_{d_1}^{\overline{F}}} f_q^{\overline{F}}(x_{l_{d_1}}).$$

On the other hand we have that

$$\sum_{q \in E_{d_1}^{\overline{F}}} f_q^{\overline{F}}(x_{l_{d_1}}) \leq |supp(x_{l_{d_1}})| \cdot \sum_{q \in E_{d_1}^{\overline{F}}} \frac{1}{w(f_q^{\overline{F}})} \leq \frac{|supp(x_{l_{d_1}})|}{w(\phi_{\overline{F}})} < \frac{|supp(x_{l_{d_1}})|}{C} = \delta$$

a contradiction.

Now applying twice Ramsey's theorem for the set $[L_{1,d_1}]^{k-d_1}$ we obtain $L_2 \in [L_{1,d_1}]$, $j_0 \in \mathbb{N}$ and $r_1 < r_2 < r_3 \in \{1, \dots, n_{2j_0+1}\}$ such that

- (1) for every $F \in [L_2]^{k-d_1}$, it follows that $w(\phi_{\overline{F}}) = m_{2j_0+1}$
- (2) for every $F \in [L_2]^{k-d_1}$ and every $i \in \{1, 2, 3\}$, we have that $E_{d_i}^F = E_{d_i}$
- (3) $r_1 \in E_{d_1}$, $r_2 = \min E_{d_2}$, $r_3 \in E_{d_3}$ and for every $F = \{n_{d_1+1} < \dots < n_k\} \in [L_2]^{k-d_1}$, then
 - i. $f_{r_1}^{\overline{F}}(x_{l_{d_1}}) > \frac{\delta}{n_{2j_0+1}}$, $f_{r_3}^{\overline{F}}(x_{n_{d_3}}) > \frac{\delta}{n_{2j_0+1}}$ and
 - ii. $f_{r_1}^{\overline{F}} < f_{r_2}^{\overline{F}} < f_{r_3}^{\overline{F}}$.

We assume without loss of generality that $L_2 = L_{1,d_1}$. Hence $L_2 = \{l_{d_1+1} < \dots < l_{d_2-1} < \dots\}$. Let $L_{2,d_2-1} = L_2 \setminus \{l_{d_1+1} < \dots < l_{d_2-1}\}$. Moreover if $F \in [L_{2,d_2-1}]^{k-(d_2-1)}$, we denote by \overline{F} the set $\overline{F} = \{l_1 < \dots < l_{d_1} < \dots < l_{d_2-1}\} \cup F$.

We consider the following sets

$$A = \{F = \{n_{d_2} < \dots < n_k\} \in [L_{2,d_2-1}]^{k-(d_2-1)} : \#E_{n_{d_2}}^{\phi_{\overline{F}}} \geq 2\} \text{ and}$$

$$B = \{F = \{n_{d_2} < \dots < n_k\} \in [L_{2,d_2-1}]^{k-(d_2-1)} : \#E_{n_{d_2}}^{\phi_{\overline{F}}} = 1\}.$$

It is obvious that $[L_{2,d_2-1}]^{k-(d_2-1)} = A \cup B$. Hence from Ramsey's theorem we may assume that $[L_{2,d_2-1}]^{k-(d_2-1)} \subset A$ or $[L_{2,d_2-1}]^{k-(d_2-1)} \subset B$. We distinguish the following cases.

Case 1. Let $[L_{2,d_2-1}]^{k-(d_2-1)} \subset A$.

Let $p \in \mathbb{N}$ such that $p > \frac{M \cdot n_{2j_0+1}}{\delta}$.

We consider vectors $y_1 < \dots < y_{p^2}$ such that $y_i \in \{x_n : n \in L_{2,d_2-1}\}$ for all $i = 1, \dots, p^2$ with $x_{l_{d_2-1}} < y_1$. We set

$$y_i = x_{n_{d_2(i)}}, i = 1, \dots, p^2, \text{ where } n_{d_2(1)} < \dots < n_{d_2(p^2)} \text{ in } L_{2,d_2-1} \text{ and } l_{d_2-1} < n_{d_2(1)}.$$

We also fix vectors $x_{n_{d_2+1}} < \dots < x_{n_{d_3}} < \dots < x_{n_k}$ such that

$$n_{d_2+1} < \dots < n_{d_3} < \dots < n_k \text{ in } L_{2,d_2-1} \text{ and } y_{p^2} < x_{n_{d_2+1}}.$$

For every $i = 1, \dots, p^2$ we consider the following subsets of $[L_{2,d_2-1}]^{k-(d_2-1)}$

$$F_i = \{n_{d_2(i)} < n_{d_2+1} < \dots < n_k\}.$$

For every $i = 1, \dots, p^2$ there exists $f_{r_3}^{\overline{F}_i} \in G_0$ with even weight such that $f_{r_3}^{\overline{F}_i}(x_{n_{d_3}}) > \frac{\delta}{n_{2j_0+1}}$.

Since $\#E_{n_{d_2(i)}}^{\phi_{\overline{F}_i}} \geq 2$ for all $i \in \{1, \dots, p^2\}$, it follows that the functionals $\{f_{r_2}^{\overline{F}_i} : i = 1, \dots, p^2\}$, are successive. Hence the functionals $\{f_{r_3}^{\overline{F}_i} : i = 1, \dots, p^2\}$, have pairwise different weights.

We consider the functional $f = \sum_{i=1}^{p^2} \frac{1}{p} f_{r_3}^{\overline{F}_i}$ which obviously belongs to G_0 . Hence

$$M < p \cdot \frac{\delta}{n_{2j_0+1}} \leq f(x_{n_{d_3}}) \leq \|x_{n_{d_3}}\|_{G_0} \leq M$$

, a contradiction.

Case 2. Let $[L_{2,d_2-1}]^{k-(d_2-1)} \subset B$.

We consider the following sets:

$C = \{\{q_1 < q_2\} \in [L_{2,d_2-1}]^2 : \text{if } F_1 = \{q_1 < n_{d_2+1} < \dots < n_k\}, F_2 = \{q_2 < n_{d_2+1} < \dots < n_k\} \text{ subsets of } [L_{2,d_2-1}]^{k-(d_2-1)}, \text{ then } \overline{f_{r_2}^{F_1}} = \overline{f_{r_2}^{F_2}}\} \text{ and } D = [L_{d_2-1}]^2 \setminus C.$

From Ramsey's Theorem we assume without loss of generality that $[L_{d_2-1}]^2 \subset C$ or $[L_{d_2-1}]^2 \subset D$. We distinguish the following cases.

Case 2.1. Let $[L_{2,d_2-1}]^2 \subset D$. In this case we derive to contradiction following the same steps as in case 1.

Case 2.2. Let $[L_{2,d_2-1}]^2 \subset C$.

In this case we will prove that $\ell^1(\mathbb{N})$ embeds isomorphically in the space \mathfrak{X}_{G_0} , which contradicts to the fact that \mathfrak{X}_{G_0} is reflexive.

We will prove that the sequence $(x_q)_{q \in L_{2,d_2-1}}$ is equivalent to the usual basis of $\ell^1(\mathbb{N})$.

Let $n \in \mathbb{N}$ and a_0, \dots, a_n real numbers. Since $L_{2,d_2-1} = \{l_{d_2+i}, i = 0, 1, \dots\}$ we get that $\|\sum_{i=0}^n a_i x_{l_{d_2+i}}\|_{G_0} \leq M \sum_{i=0}^n |a_i|$.

We consider the following elements of $[L_{2,d_2-1}]^{k-(d_2-1)}$

$$F_q = \{q < n_{d_2+1} < \dots < n_k\}, q \in L_{2,d_2-1}.$$

From the fact that $[L_{2,d_2-1}]^2 \subset C$, we have that $\overline{f_{r_2}^{F_{q_1}}} = \overline{f_{r_2}^{F_{q_2}}} = f$ for all $q_1, q_2 \in L_{2,d_2-1}$. Moreover, since $\#E_q^{\phi_{F_q}} = 1$ for all $q \in L_{2,d_2-1}$, we obtain that $E_{d_2} = E_{d_2}^{\overline{F_q}} = \{r_2\}$. Hence from (2.1) we have that $\overline{f_{r_2}^{F_q}}(x_q) > \delta$ for all $q \in L_{2,d_2-1}$ and thus

$$\|\sum_{i=0}^n a_i x_{l_{d_2+i}}\|_{G_0} \geq f(\sum_{i=0}^n a_i x_{l_{d_2+i}}) \geq \sum_{i=0}^n a_i f(x_{l_{d_2+i}}) \geq \delta \sum_{i=0}^n a_i$$

and from the 1-unconditionality of the basis of the space \mathfrak{X}_{G_0} we obtain that

$$\|\sum_{i=0}^n a_i x_{l_{d_2+i}}\|_{G_0} \geq \delta \sum_{i=0}^n |a_i|.$$

The following is an immediate consequence of the previous lemma.

Corollary 2.1. *Let $k \in \mathbb{N}$, $\delta > 0$ and $(x_n)_{n \in \mathbb{N}}$ be a block sequence which is $\|\cdot\|_{W_0}$ bounded. Then for every $L \in [\mathbb{N}]$, there exists $M \in [L]$ such that for every $n_1 < \dots < n_k$ in M the block finite sequence $(x_{n_i})_{i=1}^k$ is not δ -separated.*

Lemma 2.2. *Let $k, j_0 \in \mathbb{N}, \delta > 0, (x_n)_{n=1}^k$ be a finite block sequence and $\phi \in (W_0 \setminus G_0)$ of the form $\phi = \frac{1}{m_{2j_0+1}} \sum_{i=1}^q f_i$, where $q \leq n_{2j_0+1}$ and $(f_i)_{i=1}^q$ successive elements in G_0 such that*

- (1) $(x_n)_{n=1}^k$ is not (ϕ, δ) separated and
- (2) $\delta < (\sum_{i=1}^q f_i)(x_n)$ for all $n = 1, \dots, k$.

Then there exist at most four $d_1 < d_2 < d_3 < d_4$ in $\{1, \dots, k\}$ such that $d_1 = 1, d_4 = k$ and at most three $i_1 < i_2 < i_3$ in $\{1, \dots, q\}$ such that setting

$$I_s = \{n \in \{1, \dots, k\} : d_s < n < d_{s+1}\} \text{ for } s = 1, 2, 3$$

it follows that

$$\phi(x_n) = \frac{1}{m_{2j_0+1}} f_{i_s}(x_n) \text{ for every } s = 1, 2, 3 \text{ and every } n \in I_s.$$

Hence

$$\phi(x_1 + \dots + x_k) = \frac{1}{m_{2j_0+1}} \sum_{s=1}^3 f_{i_s}(x_1 + \dots + x_k) + \sum_{i=1}^4 \phi(x_{d_i}).$$

Proof: Since $(x_n)_{n=1}^k$ is not (ϕ, δ) separated, we may assume without loss of generality that there exist exactly two $r_1 < r_2$ in $\{1, \dots, k\}$ and $E_{r_1} < E_{r_2}$ subintervals of $\{1, \dots, q\}$ such that $\delta < \sum_{p \in E_{r_i}} f_p(x_{r_i})$ for all $i = 1, 2$.

If there exists only one the proof is similar.

From the fact that $\delta < (\sum_{i=1}^q f_i)(x_n)$ for all $n = 1, \dots, k$, we obtain that

$$E_{r_1} \cap E_1^\phi \neq \emptyset \text{ and } E_{r_2} \cap E_k^\phi \neq \emptyset.$$

Setting

$$A = \{d \in \{1, \dots, k\} : \#E_d^\phi \geq 2\}$$

it is not hard to see that $\#A \leq 4$. We assume without loss of generality that $\#A = 4$. Then $A = \{d_1 = 1 < r_1 < r_2 < d_4 = k\}$. We set $d_1 = 1$, $d_2 = r_1$, $d_3 = r_2$ and $d_4 = k$.

Moreover setting

$$I_s = \{n \in \{1, \dots, k\} : d_s < n < d_{s+1}\} \text{ for } s = 1, 2, 3$$

we observe that the set

$$B = \{d \in \{1, \dots, k\} : \#E_d^\phi = 1\} = \bigcup_{s=1}^3 I_s.$$

Therefore for every $s = 1, 2, 3$ there exists $i_s \in \{1, \dots, q\}$ with $i_1 < i_2 < i_3$ such that the conclusion of lemma is satisfied.

Lemma 2.3. Let $k \in \mathbb{N}$, $\delta > 0$ with $\sqrt{k} > \frac{4}{\delta}$ and $j_0 \in \mathbb{N}$. Let also $(x_n)_{n=1}^k$ be a finite block sequence with $\|x_n\|_{W_0} \leq 1$ for all $n = 1, \dots, k$ and $\phi \in (W_0 \setminus G_0)$ with $w(\phi) = m_{2j_0+1}$ such that

- (1) $2\delta < \|\frac{x_1 + \dots + x_k}{k}\|_{G_0}$ and
- (2) $(x_n)_{n=1}^k$ is not (ϕ, δ) separated.

Then

$$\phi\left(\frac{x_1 + \dots + x_k}{k}\right) \leq \left(\frac{4}{m_{2j_0+1}} + \frac{1}{2\sqrt{k}}\right) \left\|\frac{x_1 + \dots + x_k}{k}\right\|_{G_0}.$$

Proof: Let ϕ be of the form $\phi = \frac{1}{m_{2j_0+1}} \sum_{i=1}^q f_i$, where $q \leq n_{2j_0+1}$ and $(f_i)_{i=1}^q$ successive elements in G_0 . Let

$$D = \{n \in \{1, \dots, k\} : \delta < (\sum_{i=1}^q f_i)(x_n)\}$$

and

$$D_0 = \{1, \dots, k\} \setminus D.$$

Since $(x_n)_{n=1}^k$ is not (ϕ, δ) separated, we get that $(x_n)_{n \in D}$ is also not (ϕ, δ) separated. Next, using Lemma 2.2 we will estimate the real number $\phi(\sum_{i=1}^k x_i)$. We have that

$$\begin{aligned} \phi\left(\sum_{i=1}^k x_i\right) &= \phi\left(\sum_{i \in D_0} x_i\right) + \phi\left(\sum_{i \in D} x_i\right) \leq \frac{\delta \cdot k}{m_{2j_0+1}} + \frac{1}{m_{2j_0+1}} \sum_{s=1}^3 f_{i_s}(x_1 + \dots + x_k) + \sum_{i=1}^4 \phi(x_{d_i}) \\ &\leq \frac{1}{m_{2j_0+1}} \left\|\sum_{i=1}^k x_i\right\|_{G_0} + \frac{3}{m_{2j_0+1}} \left\|\sum_{i=1}^k x_i\right\|_{G_0} + 4 \leq \frac{4}{m_{2j_0+1}} \left\|\sum_{i=1}^k x_i\right\|_{G_0} + 4 \end{aligned}$$

Hence

$$\begin{aligned}
\phi\left(\frac{x_1 + \dots + x_k}{k}\right) &= \frac{4}{m_{2j_0+1}} \left\| \frac{x_1 + \dots + x_k}{k} \right\|_{G_0} + \frac{4}{\sqrt{k}} \cdot \frac{1}{\sqrt{k}} \leq \frac{4}{m_{2j_0+1}} \left\| \frac{x_1 + \dots + x_k}{k} \right\|_{G_0} + \frac{\delta}{\sqrt{k}} \\
&\leq \frac{4}{m_{2j_0+1}} \left\| \frac{x_1 + \dots + x_k}{k} \right\|_{G_0} + \frac{1}{2\sqrt{k}} \left\| \frac{x_1 + \dots + x_k}{k} \right\|_{G_0} \\
&= \left(\frac{4}{m_{2j_0+1}} + \frac{1}{2\sqrt{k}} \right) \left\| \frac{x_1 + \dots + x_k}{k} \right\|_{G_0}.
\end{aligned}$$

The previous result yields immediately the following.

Corollary 2.2. *Let $k \in \mathbb{N}$ and $\delta > 0$ with $\sqrt{k} > \frac{4}{\delta}$. Let also $(x_n)_{n=1}^k$ be a finite block sequence with $\|x_n\|_{W_0} \leq 1$ for all $n = 1, \dots, k$ and $\phi \in (W_0 \setminus G_0)$ such that*

$$2\delta < \left\| \frac{x_1 + \dots + x_k}{k} \right\|_{G_0} < \phi\left(\frac{x_1 + \dots + x_k}{k}\right).$$

Then $(x_n)_{n=1}^k$ is (ϕ, δ) separated.

Proposition 2.1. *For every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ with $k > n$ and every block sequence $(x_n)_{n \in \mathbb{N}}$ with*

$$0 < \epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

there exists an $L \in [\mathbb{N}]$ such that

for every $n_1 < \dots < n_k$ in L and every $\phi \in (W_0 \setminus G_0)$ it follows that

$$\phi\left(\frac{x_{n_1} + \dots + x_{n_k}}{k}\right) \leq \left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{G_0}.$$

Hence

$$\left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{W_0} = \left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{G_0}.$$

Proof: Assume that the conclusion of the proposition fails. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there exist $k > n$ and a block sequence $(x_n)_{n \in \mathbb{N}}$ with

$$\epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

such that for every $L \in [\mathbb{N}]$, there exist $n_1 < \dots < n_k$ in L and $\phi \in (W_0 \setminus G_0)$ with

$$\left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{G_0} < \phi\left(\frac{x_{n_1} + \dots + x_{n_k}}{k}\right).$$

There exist k, j_k in \mathbb{N} such that

- (1) $m_{2j_k} < \sqrt{k} < k < n_{2j_k}$
- (2) $\frac{8}{\epsilon} < \frac{\sqrt{k}}{m_{2j_k}}$.

and a block sequence $(x_n)_{n \in \mathbb{N}}$ satisfying the above properties. Hence from Ramsey's theorem there exists $L \in [\mathbb{N}]$ such that for every $n_1 < \dots < n_k$ in L there exists $\phi \in (W_0 \setminus G_0)$ with

$$(2.2) \quad \left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{G_0} < \phi\left(\frac{x_{n_1} + \dots + x_{n_k}}{k}\right).$$

We observe from the choice of $k \in \mathbb{N}$, that for every $n_1 < \dots < n_k$ in L we obtain

$$(2.3) \quad \frac{\epsilon}{m_{2j_k}} < \left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{G_0}.$$

From (2.2), (2.3) and Corollary 2.2 we get that for every $F \in [L]^k$, the sequence $(x_n)_{n \in F}$ is $\frac{\epsilon}{m_{2j_k}}$ separated. Thus from Corollary 2.1, we derive to contradiction.

Proposition 2.2. Let $\epsilon > 0$ and $(x_n)_{n \in \mathbb{N}}$ be a block sequence such that

$$\epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Then for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with $k > n$ and there exist $y_1 < \dots < y_k$ in $\langle x_n : n \in \mathbb{N} \rangle$ such that

- (1) $\|y_i\|_{W_0} \leq 1$ for all $i = 1, \dots, k$ and
- (2) if $y = \frac{1}{k}(y_1 + \dots + y_k)$ then $\frac{1}{2} < \|y\|_{G_0} = \|y\|_{W_0}$.

Proof: Assume that the conclusion fails. Then there exists $n \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ with $k > n$ and every $y_1 < \dots < y_k$ in $\langle x_n : n \in \mathbb{N} \rangle$ with $\|y_i\|_{W_0} \leq 1$ for all $i = 1, \dots, k$ it follows that $\|\frac{1}{k}(y_1 + \dots + y_k)\|_{G_0} \leq \frac{1}{2}$ or $\|\frac{1}{k}(y_1 + \dots + y_k)\|_{G_0} < \|\frac{1}{k}(y_1 + \dots + y_k)\|_{W_0}$. It is obvious that there exists $k_0 \in \mathbb{N}$ with $k_0 > n$ such that for every $k \in \mathbb{N}$ with $k > k_0$ the conclusion of Proposition 2.1 is valid.

We choose $j, s \in \mathbb{N}$ such that $m_{2j} \leq \epsilon \cdot 2^{s-1} < 2^s \leq k_0^s \leq n_{2j}$.

We set $z_n^{(1)} = x_n, n \in \mathbb{N}$. Then from Proposition 2.1 there exists an $L_1 \in [\mathbb{N}]$ such that for every $n_1 < \dots < n_{k_0}$ in L_1 it follows that

$$(2.4) \quad \left\| \frac{1}{k_0}(z_{n_1}^{(1)} + \dots + z_{n_{k_0}}^{(1)}) \right\|_{G_0} = \left\| \frac{1}{k_0}(z_{n_1}^{(1)} + \dots + z_{n_{k_0}}^{(1)}) \right\|_{W_0}.$$

Let $L_1 = \{l_1^{(1)} < \dots < l_i^{(1)} < l_{i+1}^{(1)} < \dots\}$. We set

$$w_{1,n} = \sum_{i=(n-1)k_0+1}^{nk_0} x_{l_i^{(1)}}, n \in \mathbb{N}.$$

Let $n \in \mathbb{N}$. Since $\|z_n^{(1)}\|_{W_0} \leq 1$ and (2.4) holds, we obtain that $\|\frac{1}{k_0}w_{1,n}\|_{G_0} = \|\frac{1}{k_0}w_{1,n}\|_{W_0} \leq \frac{1}{2}$. Hence $\|\frac{2}{k_0}w_{1,n}\|_{W_0} \leq 1$.

We set $z_n^{(2)} = \frac{2}{k_0}w_{1,n}, n \in \mathbb{N}$. Then from Proposition 2.1 there exists an $L_2 \in [\mathbb{N}]$ such that for every $n_1 < \dots < n_{k_0}$ in L_2 it follows that

$$(2.5) \quad \left\| \frac{1}{k_0}(z_{n_1}^{(2)} + \dots + z_{n_{k_0}}^{(2)}) \right\|_{G_0} = \left\| \frac{1}{k_0}(z_{n_1}^{(2)} + \dots + z_{n_{k_0}}^{(2)}) \right\|_{W_0}.$$

Let $L_2 = \{l_1^{(2)} < \dots < l_i^{(2)} < l_{i+1}^{(2)} < \dots\}$. We set

$$w_{2,n} = \sum_{i=(n-1)k_0+1}^{nk_0} w_{1,l_i^{(2)}}, n \in \mathbb{N}.$$

It is obvious that for every $n \in \mathbb{N}$ the vector $w_{2,n}$ consists of k_0^2 blocks of the sequence $(x_n)_{n \in L_1}$. Moreover, since for a $n \in \mathbb{N}$ we have that $\|\frac{2}{k_0}w_{1,n}\|_{W_0} \leq 1$, we get from (2.5) that $\|\frac{2}{k_0^2}w_{2,n}\|_{G_0} = \|\frac{2}{k_0^2}w_{2,n}\|_{W_0} \leq \frac{1}{2}$. Hence $\|\frac{2^2}{k_0^2}w_{2,n}\|_{W_0} \leq 1$ for all $n \in \mathbb{N}$.

In this way we inductively construct

- (1) a finite sequence $(L_r)_{r=1}^s$ of infinite subsets of \mathbb{N} , where $L_r = \{l_1^{(r)} < \dots < l_i^{(r)} < l_{i+1}^{(r)} < \dots\}, r = 1, \dots, s$ and
- (2) for every $r = 1, \dots, s$ a sequence $(w_{r,n})_{n \in \mathbb{N}}$ of the form $w_{r,n} = \sum_{i=(n-1)k_0+1}^{nk_0} w_{r-1,l_i^{(r)}}, n \in \mathbb{N}, r = 2, \dots, s$ where $w_{1,n} = \sum_{i=(n-1)k_0+1}^{nk_0} x_{l_i^{(1)}}, n \in \mathbb{N}$ such that

$$\|w_{r,n}\|_{G_0} = \|w_{r,n}\|_{W_0} \leq \frac{k_0^r}{2^r} \text{ for } r = 1, \dots, s.$$

We can see that if $r \in \{1, \dots, s\}$ then for every $n \in \mathbb{N}$ the vector $w_{r,n}$ consists of k_0^r block vectors of the sequence $(x_n)_{n \in L_1}$.

Let $n_0 \in \mathbb{N}$ and $w_{s,n_0} = \sum_{i=1}^{k_0^s} x_{l_{q_i}}^{(1)}$. For every $i = 1, \dots, k_0^s$ there exists $g_i \in G_0$ with $\text{ran}(g_i) \subset \text{ran}(x_{l_{q_i}}^{(1)})$ such that $g_i(x_{l_{q_i}}^{(1)}) > \frac{\epsilon}{2}$. We consider the functional $g = \frac{1}{m_{2j}} \sum_{i=1}^{k_0^s} g_i$ which belongs to G_0 . Then

$$\frac{\epsilon \cdot k_0^s}{2m_{2j}} < g(w_{s,n_0}) \leq \frac{k_0^s}{2^s}$$

which yields that $2^{s-1} < m_{2j}$, a contradiction.

3. Rapidly increasing sequences in \mathfrak{X}_{G_0} and in \mathfrak{X}_0

We begin with the definition of the Rapidly increasing Sequences(RIS) in \mathfrak{X}_{G_0} and in \mathfrak{X}_0 and the definition of $M - \ell_k^1$ averages in \mathfrak{X}_0 .

Definition 3.1. (*Rapidly increasing sequences in $\mathfrak{X}_{G_0}, \mathfrak{X}_0$*) Let $(x_n)_{n \in \mathbb{N}}$ be a block sequence and C, ϵ positive numbers. This sequence will be called

(i.) (C, ϵ) RIS in \mathfrak{X}_{G_0} if the following hold:

1. $\|x_n\|_{G_0} \leq C$ for all $n \in \mathbb{N}$.
2. There exists a strictly increasing sequence of natural numbers $(j_n)_{n \in \mathbb{N}}$ such that $\frac{|\text{supp } x_n|}{m_{j_n+1}} < \epsilon$, for all $n \in \mathbb{N}$.
3. For every $n \in \mathbb{N}$ and every $f \in G_0$ with $w(f) = m_{2i} < m_{j_n}$ we have that $|f(x_n)| \leq \frac{C}{m_{2i}}$.

(ii.) (C, ϵ) RIS in \mathfrak{X}_0 if the following hold:

1. $\|x_n\|_{W_0} \leq C$ for all $n \in \mathbb{N}$.
2. There exists a strictly increasing sequence of natural numbers $(j_n)_{n \in \mathbb{N}}$ such that $\frac{|\text{supp } x_n|}{m_{j_n+1}} < \epsilon$, for all $n \in \mathbb{N}$.
3. For every $n \in \mathbb{N}$ and $f \in W_0$ with $w(f) = m_i < m_{j_n}$ we have that $|f(x_n)| \leq \frac{C}{m_i}$.

Every sequence $(j_n)_{n \in \mathbb{N}}$ of natural numbers like in this definition is said to be the associated sequence of the RIS $(x_n)_{n \in \mathbb{N}}$.

Next, we define the $M - \ell_k^1$ averages in \mathfrak{X}_0 .

Definition 3.2. (ℓ_k^1 - averages in \mathfrak{X}_0) Let $k \in \mathbb{N}$, $M > 0$ and $(e_n)_{n \in \mathbb{N}}$ the Schauder basis of \mathfrak{X}_0 . A vector $x \in \langle e_n : n \in \mathbb{N} \rangle$ is said to be a $M - \ell_k^1$ average in \mathfrak{X}_0 if:

1. $\|x\|_{G_0} = \|x\|_{W_0} > \frac{1}{2}$.
2. There exists $x_1 < \dots < x_k$ in $\langle e_n : n \in \mathbb{N} \rangle$ with $\|x_i\|_{W_0} \leq M$ for all $i = 1, \dots, k$ such that $x = \frac{1}{k} \sum_{i=1}^k x_i$.

In the following Lemma, according to Proposition 2.2, it is easily checked that in every block subspace $\langle x_n, n \in \mathbb{N} \rangle$, where

$$0 < \epsilon \leq \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

there exists a block sequence of $(x_n)_{n \in \mathbb{N}}$ of l^1 -averages with increasing lengths. A consequence of this result is the existence of a $(3, \delta)$ RIS in \mathfrak{X}_0 for a fixed $\delta > 0$. The proof of the existence of the RIS follows the lines of the proof of Proposition II.25 in [ATO].

Lemma 3.1. Let $\epsilon, \delta > 0$ and $(x_n)_{n \in \mathbb{N}}$ be a block sequence such that

$$\epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Then there exist a block sequence $(y_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and a subsequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that

- i. For every $n \in \mathbb{N}$ the vector y_n is a $1 - \ell_{k_n}^1$ average in \mathfrak{X}_0 .

- ii. The sequence $(y_n)_{n \in \mathbb{N}}$ is $(3, \delta)$ RIS in \mathfrak{X}_0 .

Proof: The first assertion is an immediate consequence of Proposition 2.2. The proof of the second follows the lines of the proof of Proposition II.25 in [ATO].

4. The existence of exact pairs in \mathfrak{X}_0

The aim of this section is the existence of exact pairs and dependent sequences in \mathfrak{X}_0 . In order to achieve this we use the Basic Inequality for the space \mathfrak{X}_0 . As usually we will use an auxiliary space F_{j_0} , $j_0 \in \mathbb{N}$ and our approach follows similar steps as in [AMP].

Definition 4.1. Let $j_0 \in \mathbb{N}$ with $j_0 > 1$. We denote by F_{j_0} the minimal subset of $c_{00}(\mathbb{N})$ such that:

- (1) $C_{j_0} = \{ \sum_{i \in F} \epsilon_i e_i^* : |\epsilon_i| = 1, \#F \leq n_{j_0-1} \} \subset F_{j_0}$.
- (2) F_{j_0} is closed under the operation $(\mathcal{A}_{2n_{2j}}, \frac{1}{m_{2j}})$ for every $j \in \mathbb{N}$.
- (3) For every A, B nonempty finite subsets of \mathbb{N} with $A \cap B = \emptyset$, for every $(a_i)_{i \in A \cup B}$ finite sequence of real numbers such that $\sum_{i \in A \cup B} a_i^2 \leq 1$, for every $(f_i)_{i \in A}$, where $f_i \in F_{j_0}$ is a result of a $(\mathcal{A}_{2n_{j_i}}, \frac{1}{m_{j_i}})$ operation with $w(f_{i_1}) \neq w(f_{i_2})$ for all $i_1 \neq i_2 \in A$, and for every finite sequence of natural numbers $(t_i)_{i \in B}$, where $t_i \neq t_j$ for all $i \neq j \in B$, it follows that $(\sum_{i \in A} a_i f_i + \sum_{i \in B} a_i e_{t_i}) \in F_{j_0}$.

We notice that for an $f \in F_{j_0}$ we say that f has weight m_{2j} or that is a result of a $(\mathcal{A}_{2n_{2j}}, \frac{1}{m_{2j}})$ operation and we write $w(f) = m_{2j}$ if and only if there exists $d \in \mathbb{N}$ with $d \leq 2n_{2j}$ and $f_1 < \dots < f_d \in F_{j_0}$ such that $f = \frac{1}{m_{2j}} \sum_{i=1}^d f_i$.

We also define the auxiliary space F'_{j_0} to be the following subset of $c_{00}(\mathbb{N})$.

$$F'_{j_0} = F_{j_0} \cup \left\{ \frac{1}{m_{2j+1}} \sum_{i=1}^d f_i : j \in \mathbb{N}, d \leq 2n_{2j+1}, f_i \in F_{j_0} \right\}.$$

The weight of a functional f belonging to $F'_{j_0} \setminus F_{j_0}$ is defined in a similar way as for the functionals of F_{j_0} .

We begin with the estimations of a functional belonging to the auxiliary space, acting in averages of the basis of length n_j . For a proof of the following Lemma we refer to Lemma 8.10 in [AMP].

Lemma 4.1. There exists $M > 0$ such that if $j_0 \in \mathbb{N}$ then

1. if $f \in F'_{j_0}$ with $w(f) = m_i, i \in \mathbb{N}$ and $k_1 < \dots < k_{n_{j_0}}$ natural numbers then

$$\left| f\left(\frac{1}{n_{j_0}} \sum_{r=1}^{n_{j_0}} e_{k_r}\right) \right| \leq \begin{cases} \frac{4}{m_i m_{j_0}}, & \text{if } i < j_0 \\ \frac{M}{m_i}, & \text{if } i \geq j_0 \end{cases}$$

2. if $f \in F'_{2j_0+1}$ with $w(f) = m_i, i \neq 2j_0 + 1$ and $k_1 < \dots < k_{n_{2j_0+1}}$ natural numbers then

$$\left| f\left(\frac{1}{n_{2j_0+1}} \sum_{r=1}^{n_{2j_0+1}} e_{k_r}\right) \right| \leq \begin{cases} \frac{4}{m_i m_{2j_0+1}}, & \text{if } i < 2j_0 + 1 \\ \frac{M}{m_i}, & \text{if } i \geq 2j_0 + 1 \end{cases}$$

The proof of the following proposition (basic inequality in \mathfrak{X}_0) follows the lines of the proof of Proposition 9.3 in [AMP].

Proposition 4.1. (basic inequality in \mathfrak{X}_0) Let $(x_k)_{k \in \mathbb{N}}$ be a (C, ϵ) RIS in \mathfrak{X}_0 where $C > 0, \epsilon > 0$ and $j_0 \in \mathbb{N}$. Let also $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of real numbers.

Then for every $f \in W_0$ and every finite interval I of \mathbb{N} there exist a functional $g \in F'_{j_0}$ and a real number $\epsilon_f \leq \epsilon$ such that

$$|f(\sum_{k \in I} \lambda_k x_k)| \leq C(g(\sum_{k \in I} |\lambda_k| e_k) + \epsilon_f \cdot \sum_{k \in I} |\lambda_k|).$$

If we assume that $w(f) = m_{2j}$ then $g = 0$ or $g \in \{\pm e_n^* : n \in \mathbb{N}\}$ or $w(g) = w(f)$ and $\epsilon_f \leq \frac{\epsilon}{w(f)}$.

Remark 4.1. The Basic Inequality in \mathfrak{X}_{G_0} is analogous to that in \mathfrak{X}_0 . Specifically, for a block (C, ϵ) RIS in \mathfrak{X}_{G_0} and a functional $f \in G_0$, there exist a functional $g \in F_{j_0}$ and a real number $\epsilon_f \leq \epsilon$ satisfying the same properties as for the basic inequality in \mathfrak{X}_0 .

The proofs of the following two propositions (Proposition 4.2 and 4.3) are based in Basic Inequality in \mathfrak{X}_0 and Lemma 4.1. For a proof we refer to Proposition 9.4 in [AMP].

Proposition 4.2. Let $j_0 \in \mathbb{N}$ and $(x_n)_{n \in \mathbb{N}}$ be a (C, ϵ) RIS in \mathfrak{X}_0 with $(j_n)_{n \in \mathbb{N}}$ its associated sequence such that $C > 0$, $0 < \epsilon < \frac{2}{m_{j_0}^2}$ and $j_1 > j_0$. Let also $k_1 < \dots < k_{n_{j_0}}$ natural numbers and $f \in W_0$ with $w(f) = m_i$. Then

$$\begin{aligned} \text{i. } |f(\frac{x_{k_1} + \dots + x_{k_{n_{j_0}}}}{n_{j_0}})| &\leq \begin{cases} \frac{5C}{m_i m_{j_0}}, & \text{if } i < j_0 \\ \frac{MC}{m_i} + \frac{2C}{m_{j_0}^2}, & \text{if } i \geq j_0 \end{cases} \\ \text{ii. } \|\frac{x_{k_1} + \dots + x_{k_{n_{j_0}}}}{n_{j_0}}\|_{W_0} &\leq \frac{3C}{m_{j_0}}. \end{aligned}$$

M is the positive number appearing in Lemma 4.1.

Remark 4.2. For a (C, ϵ) RIS in \mathfrak{X}_{G_0} we have the same estimations for every functional f belonging to G_0 and the $\|\cdot\|_{G_0}$ norm as in Proposition 4.2.

Proposition 4.3. Let $j_0 \in \mathbb{N}$ and $(x_n)_{n \in \mathbb{N}}$ be a (C, ϵ) RIS in \mathfrak{X}_{G_0} with $(j_n)_{n \in \mathbb{N}}$ its associated sequence such that $C > 0$, $0 < \epsilon < \frac{2}{m_{2j_0+1}^2}$ and $j_1 > 2j_0 + 1$. Let also $k_1 < \dots < k_{n_{2j_0+1}}$ natural numbers and $|a_i| \leq 1$ for all $i = 1, \dots, n_{2j_0+1}$. Then

$$\|\frac{a_1 x_{k_1} + \dots + a_{n_{2j_0+1}} x_{k_{n_{2j_0+1}}}}{n_{2j_0+1}}\|_{G_0} \leq \frac{3C}{m_{2j_0+1}^2}.$$

Next, we define the exact pairs in \mathfrak{X}_0 .

Definition 4.2. (*exact pairs in \mathfrak{X}_0*) Let $x \in \mathfrak{X}_0$ with finite support and $f \in G_0$. The pair (x, f) is called a $(C, 2j, \theta)$ exact pair in \mathfrak{X}_0 , where $C \geq 1$, $j \in \mathbb{N}$ and $\theta \geq 0$ if the following hold:

- (i) $\frac{1}{2} \leq \|x\|_{G_0} = \|x\|_{W_0} \leq C$
- (ii) $w(f) = m_{2j}$
- (iii) $f(x) = \theta$
- (iv) If $g \in W_0$ with $w(g) = m_i$, then $|g(x)| \leq \frac{C}{m_i}$ if $i < 2j$ and $|g(x)| \leq \frac{C}{m_{2j}}$, if $i > 2j$.

In the following Proposition, using Proposition 2.1, Lemma 3.1 and Basic Inequality in \mathfrak{X}_0 , we get that in every block subspace $\langle x_n, n \in \mathbb{N} \rangle$, where

$$0 < \epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

there exist an exact pair in \mathfrak{X}_0 . The proof follows the lines of the proof of Proposition II.32 in [ATO] or Proposition 10.2 in [AMP].

Proposition 4.4. Let $j \in \mathbb{N}$, $\epsilon > 0$ and $(x_n)_{n \in \mathbb{N}}$ be a block sequence such that

$$\epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Then there exists a pair (x, f) with $x \in \langle x_n, n \in \mathbb{N} \rangle$ which is $(15, 2j, \frac{1}{2})$ exact pair in \mathfrak{X}_0 .

Proof: From Lemma 3.1, there exist a block sequence $(y_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and a subsequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that

- (1) for every $n \in \mathbb{N}$ the vector y_n is a $1 - \ell_{k_n}^1$ average in \mathfrak{X}_0 and
- (2) $(y_n)_{n \in \mathbb{N}}$ is $(3, \delta)$ RIS in \mathfrak{X}_0 , where $0 < \delta < \frac{2}{m_{2j}^2}$ and the first term of the associated sequence is bigger than $2j$.

Therefore if $k_1 < \dots < k_{n_{2j}}$ natural numbers, then there exist $f_1 < \dots < f_{n_{2j}}$ in G_0 such that $f_i(y_{k_i}) = \frac{1}{2}$ and $\text{ran}(f_i) \subset \text{ran}(y_{k_i})$ for all $i = 1, \dots, n_{2j}$. We set

$$x = \frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} y_{k_i} \text{ and } f = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i.$$

Then it easily checked, using Proposition 4.2, that the pair (x, f) is a $(15, 2j, \frac{1}{2})$ exact pair in \mathfrak{X}_0 .

We define the dependent sequences in \mathfrak{X}_0 .

Definition 4.3. (*dependent sequences in \mathfrak{X}_0*) Let $j \in \mathbb{N}, \theta \geq 0$ and $C \geq 1$. A finite sequence of pairs $(x_i, f_i)_{i=1}^{n_{2j+1}}$ with $x_i \in \mathfrak{X}_0$ for all $i \in \{1, \dots, n_{2j+1}\}$ is called a $(C, 2j+1, \theta)$ -dependent sequence in \mathfrak{X}_0 if the following hold:

1. $(f_i)_{i=1}^{n_{2j+1}}$ is a $\sigma - n_{2j+1}$ special sequence with $w(f_i) = m_{2j_i}$ where $2j_i = \sigma(f_1, \dots, f_{i-1}), i \in \{2, \dots, n_{2j+1}\}$ and $2j_1 \in \{2j : j \in \Omega_1\}$
2. each pair (x_i, f_i) is a $(C, 2j_i, \theta)$ exact pair in \mathfrak{X}_0
3. $\text{ran}(f_i) \cup \text{ran}(x_i) < \text{ran}(f_{i+1}) \cup \text{ran}(x_{i+1})$ for all $i = 1, \dots, n_{2j+1} - 1$.

In the following Proposition, using the existence of exact pairs in every block subspace $\langle x_n, n \in \mathbb{N} \rangle$ of \mathfrak{X}_0 , with

$$0 < \epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

we construct a dependent sequence in \mathfrak{X}_0 .

Proposition 4.5. Let $j \in \mathbb{N}, \epsilon > 0$ and $(x_n)_{n \in \mathbb{N}}$ be a block sequence such that

$$\epsilon < \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Then there exists a finite sequence of pairs $(x_i, f_i)_{i=1}^{n_{2j+1}}$ with $x_i \in \langle x_n, n \in \mathbb{N} \rangle$ for all $i = 1, \dots, n_{2j+1}$, which is a $(15, 2j+1, \frac{1}{2})$ -dependent sequence in \mathfrak{X}_0 with $\text{ran}(f_i) = \text{ran}(x_i)$ for all $i = 1, \dots, n_{2j+1}$.

Proof: It follows easily from an inductive application of Proposition 4.4.

Proposition 4.6. Let $j_0 \in \mathbb{N}, \theta \geq 0, C \geq 1$ and $(x_i, f_i)_{i=1}^{n_{2j_0+1}}$ be a $(C, 2j_0+1, \theta)$ -dependent sequence in \mathfrak{X}_0 with $\text{maxsupp}(f_i) \geq \# \text{supp}(x_i)$ for all $i = 1, \dots, n_{2j_0+1}$. Then

- (1) if $\theta = \frac{1}{2}$, it holds

$$\left\| \frac{1}{n_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} (-1)^i x_i \right\|_{W_0} \leq \frac{8C}{m_{2j_0+1}^2}.$$

- (2) if $\theta = 0$, it holds

$$\left\| \frac{1}{n_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} x_i \right\|_{W_0} \leq \frac{8C}{m_{2j_0+1}^2}.$$

Proof: The proof follows the lines of the proof of Proposition III.6 in [ATO].

5. Properties of \mathfrak{X}_0

In this section we prove that the Banach space \mathfrak{X}_0 is c_0 -saturated, i.e. every closed, infinite dimensional subspace contains an isomorphic copy of c_0 . Also it is proved that if Y, Z are closed, infinite dimensional subspaces of \mathfrak{X}_0 such that the direct sum $Y \oplus Z$ is a closed subspace, then one at least of the subspaces is embedded isomorphically into c_0 and the other contains an isomorph of \mathfrak{X}_0 . Our proof uses a deep result due to Kalton that characterizes the subspaces of c_0 . For the convenience, by a subspace we always mean a closed and infinite dimensional one.

Proposition 5.1. *The identity operator $id : \mathfrak{X}_0 \longrightarrow \mathfrak{X}_{G_0}$ is strictly singular.*

Proof: Assume on the contrary. Then there exists a block sequence $(x_n)_{n \in \mathbb{N}}$ with $\|x_n\|_{W_0} = 1, n \in \mathbb{N}$ such that the operator

$$id|_{\langle x_n, n \in \mathbb{N} \rangle} : (\langle x_n, n \in \mathbb{N} \rangle, \|\cdot\|_{W_0}) \rightarrow (\langle x_n, n \in \mathbb{N} \rangle, \|\cdot\|_{G_0})$$

is an isomorphism. Hence there exist $m > 0$ and $M > 0$ such that

$$(5.1) \quad m\|x\|_{W_0} \leq \|x\|_{G_0} \leq M\|x\|_{W_0}, \text{ for every } x \in \langle x_n, n \in \mathbb{N} \rangle.$$

Let $j \in \mathbb{N}$ with $\frac{1}{m_{2j+1}} < \frac{m}{90}$. From (5.1) we get that $m \leq \|x_n\|_{G_0} \leq \|x_n\|_{W_0} \leq 1$ for all $n \in \mathbb{N}$. Therefore from Proposition 4.5 there exists a $(15, 2j+1, \frac{1}{2})$ dependent sequence $(w_i, f_i)_{i=1}^{n_{2j+1}}$ in \mathfrak{X}_0 with $\text{ran}(f_i) = \text{ran}(w_i)$ and $w_i \in \langle x_n, n \in \mathbb{N} \rangle$ for all $i = 1, \dots, n_{2j+1}$.

We set

$$w = \frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} w_i \text{ and } f = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i.$$

It is obvious that $f \in W_0$ and from the definition of exact pairs in \mathfrak{X}_0 we get that $\|w\|_{W_0} \geq f(w) > \frac{1}{2}$.

On the other hand Proposition 4.3 yields that $\|w\|_{G_0} \leq \frac{45}{m_{2j+1}}$.

Therefore from inequality (5.1) it follows that $m \leq \frac{90}{m_{2j+1}}$, a contradiction.

Lemma 5.1. *Let $\epsilon > 0$, $(\epsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers with $\sum_{k=1}^{\infty} \epsilon_k < \epsilon$, $(j_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of even numbers and $(x_k)_{k \in \mathbb{N}}$ be a block sequence such that*

- i. $\|x_k\|_{W_0} = 1$ for all $k \in \mathbb{N}$
- ii. $\frac{|\text{supp } x_k|}{m_{j_k}} < \epsilon_k$, for all $k \in \mathbb{N}$ and
- iii. $\|x_{k+1}\|_{G_0} \leq \frac{\epsilon_k}{n_{j_k}}$ for all $k \in \mathbb{N}$.

Then $(x_k)_{k \in \mathbb{N}}$ is $(1 + \epsilon)$ -equivalent to the usual basis of $c_0(\mathbb{N})$.

Proof: Let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$ with $\max\{|a_i|, i = 1, \dots, n\} = 1$ for $i = 1, \dots, n$. We assume without loss of generality that $n \geq 2$.

Since the block sequence $(x_k)_{k \in \mathbb{N}}$ is bimonotone and normalized it follows that $\|\sum_{i=1}^n a_i x_i\|_{W_0} \geq 1$.

We will prove that $\|\sum_{i=1}^n a_i x_i\|_{W_0} \leq 1 + \epsilon$.

Let $f \in W_0$. We distinguish the following cases.

Case 1. Let $f \in G_0$. Then

$$|f(\sum_{i=1}^n a_i x_i)| \leq \sum_{i=1}^n |a_i| \cdot |f(x_i)| \leq |f(x_1)| + \sum_{i=2}^n |f(x_i)| \leq 1 + \sum_{i=2}^n \epsilon_{i-1} \leq 1 + \epsilon.$$

Case 2. Let $f \in (W_0 \setminus G_0)$.

We assume without loss of generality that f is of the form $f = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} f_i$, where $(f_i)_{i=1}^{n_{2j+1}}$ is a

$\sigma - n_{2j+1}$ special sequence. Let $j_{k-1} < 2j + 1 < j_k$ where $2 \leq k < n$. Then

$$\begin{aligned} |f(\sum_{i=1}^n a_i x_i)| &\leq |f(\sum_{i=1}^{k-1} a_i x_i)| + |f(a_k x_k)| + |f(\sum_{i=k+1}^n a_i x_i)| \leq \sum_{i=1}^{k-1} |f(x_i)| + 1 + \sum_{i=k+1}^n |f(x_i)| \\ &\leq 1 + \sum_{i=1}^{k-1} \frac{|supp(x_i)|}{m_{j_i}} + \sum_{i=k+1}^n n_{j_{k-1}} \cdot \|x_i\|_{G_0} = 1 + \sum_{i=1}^{k-1} \epsilon_i + \sum_{i=k+1}^n \frac{n_{j_{k-1}} \cdot \epsilon_i}{n_{j_{i-1}}} \\ &\leq 1 + \epsilon. \end{aligned}$$

Lemma 5.2. *Let $\epsilon > 0$ and $(x_k)_{k \in \mathbb{N}}$ be a block sequence such that*

- i. $\lim_{k \rightarrow \infty} \|x_k\|_{G_0} = 0$ and
- ii. $\|x_k\|_{W_0} = 1$ for all $k \in \mathbb{N}$.

Then there exists a subsequence which is $(1 + \epsilon)$ -equivalent to the usual basis of c_0 .

Proof: Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \epsilon_k < \epsilon$. We inductively construct a subsequence $(z_k)_{k \in \mathbb{N}}$ of $(x_k)_{k \in \mathbb{N}}$ and a strictly increasing sequence $(j_k)_{k \in \mathbb{N}}$ of even numbers such that

- i. $\frac{|supp z_k|}{m_{j_k}} < \epsilon_k$, for all $k \in \mathbb{N}$ and
- ii. $\|z_{k+1}\|_{G_0} \leq \frac{\epsilon_k}{n_{j_k}}$ for all $k \in \mathbb{N}$.

Therefore from Lemma 5.1, it follows that $(z_k)_{k \in \mathbb{N}}$ is $(1 + \epsilon)$ -equivalent to the usual basis of c_0 .

Corollary 5.1. *Let Y be a subspace of \mathfrak{X}_0 and $\epsilon > 0$. Then there exists a subspace of Y which is $(1 + \epsilon)$ -isomorphic to c_0 .*

Proof: We assume that Y is a block subspace of \mathfrak{X}_0 . Then from the well known gliding hump argument we get the result.

Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} \epsilon_k < \epsilon$.

Since the identity operator $id : \mathfrak{X}_0 \rightarrow \mathfrak{X}_{G_0}$ is strictly singular, there exists a block sequence $(y_k)_{k \in \mathbb{N}}$ in Y such that

- i. $\lim_{k \rightarrow \infty} \|y_k\|_{G_0} = 0$ and
- ii. $\|y_k\|_{W_0} = 1$ for all $k \in \mathbb{N}$.

Hence from Lemma 5.2 we obtain a subsequence which is $(1 + \epsilon)$ -equivalent to the usual basis of c_0 .

Corollary 5.2. *Let Y be a subspace of \mathfrak{X}_0 . Then Y contains a complemented copy of c_0 .*

Proof: Since Y is separable and contains an isomorphic copy of $c_0(\mathbb{N})$, then from Sobczyk's [SO] theorem it follows that c_0 is complemented in Y .

Proposition 5.2. *The basis $(e_n)_{n \in \mathbb{N}}$ of \mathfrak{X}_0 is shrinking, hence \mathfrak{X}_0^* is separable.*

Proof: Suppose not. Then there exist $\epsilon_0 > 0$, $x^* \in \mathfrak{X}_0^*$ with $\|x^*\| = 1$ and a block sequence $(x_n)_{n \in \mathbb{N}}$ with $\|x_n\|_{W_0} \leq 1$, $n \in \mathbb{N}$ such that $\epsilon_0 < x^*(x_n)$ for all $n \in \mathbb{N}$. We distinguish the following cases.

Case 1. There exists $\delta > 0$ such that $\delta < \|x_n\|_{G_0}$ for all $n \in \mathbb{N}$.

Proposition 2.1 yields that there exists $n_0 \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ with $k > n_0$ and every $L \in [\mathbb{N}]$ there exists $Q \in [L]$ such that for every $n_1 < \dots < n_k$ in Q we have that

$$\left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{W_0} = \left\| \frac{x_{n_1} + \dots + x_{n_k}}{k} \right\|_{G_0}.$$

Hence there exists $M \in [\mathbb{N}]$ and a block sequence $(z_k)_{k \in \mathbb{N}}$ of $(x_k)_{k \in \mathbb{N}}$ such that

(1) $z_k = \frac{1}{|F_k|} \sum_{i \in F_k} x_i$, where $F_k < F_{k+1}$ finite subsets of M and $(|F_k|)_{k \in \mathbb{N}}$ strictly increasing sequence with $|F_1| > n_0$

(2) $\epsilon_0 < x^*(z_k) < \|z_k\|_{W_0} = \|z_k\|_{G_0} \leq 1$ for all $k \in \mathbb{N}$.

Let $j \in \mathbb{N}$ with $n_{2j} > k_0$ and $\epsilon > 0$ with $\epsilon < \frac{2}{m_{2j}}$. We may assume without loss of generality that $(z_k)_{k \in \mathbb{N}}$ is $(3, \epsilon)$ RIS in \mathfrak{X}_0 such that the first term of the associated sequence is bigger than $2j$. From (2) it follows that there exist $k_1 < \dots < k_{n_{2j}}$ in M such that

$$\left\| \frac{z_{k_1} + \dots + z_{k_{n_{2j}}}}{n_{2j}} \right\|_{W_0} = \left\| \frac{z_{k_1} + \dots + z_{k_{n_{2j}}}}{n_{2j}} \right\|_{G_0}.$$

Thus from proposition 4.2 we have that

$$\left\| \frac{z_{k_1} + \dots + z_{k_{n_{2j}}}}{n_{2j}} \right\|_{W_0} = \left\| \frac{z_{k_1} + \dots + z_{k_{n_{2j}}}}{n_{2j}} \right\|_{G_0} \leq \frac{9}{m_{2j}}.$$

Since the action of x^* in every convex combination of $(z_k)_{k \in \mathbb{N}}$ is bigger than ϵ_0 , then for a sufficiently large $j \in \mathbb{N}$ we derive to contradiction.

Case 2. There exists a subsequence $(x_{p_k})_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} \|x_{p_k}\|_{G_0} = 0$.

Then from Lemma 5.1 we assume without loss of generality that $(x_{p_k})_{k \in \mathbb{N}}$ is equivalent to the usual basis of c_0 . Hence there exists $M > 0$ such that

$$k\epsilon_0 \leq x^*\left(\sum_{i=1}^k x_{p_i}\right) \leq \left\| \sum_{i=1}^k x_{p_i} \right\|_{W_0} \leq M \text{ for every } k \in \mathbb{N}$$

a contradiction.

Next we shall show a structural property for the subspaces of \mathfrak{X}_0 on which the identity operator $id : \mathfrak{X}_0 \rightarrow \mathfrak{X}_{G_0}$ is compact. Our approach uses a beautiful and deep result due to Kalton [K].

Notation 5.1. We set $[\mathbb{N}]^{<\omega} = \bigcup_{k \in \mathbb{N}} [N]^k$, where $[N]^k$ denotes the set of all finite subsets of \mathbb{N} of cardinality k . We define in $[\mathbb{N}]^{<\omega}$ the following partial order:
if $\{n_1 < \dots < n_k\}, \{m_1 < \dots < m_n\} \in [\mathbb{N}]^{<\omega}$ then

$\{n_1 < \dots < n_k\} \sqsubseteq \{m_1 < \dots < m_n\}$ if and only if $k \leq n$ and $n_i = m_i$ for all $i = 1, \dots, k$.

This partial order is called the initial segment partial order and the couple $([\mathbb{N}]^{<\omega}, \sqsubseteq)$ is a tree. A branch of this tree is identified by an infinite subset $\{q_n, n \in \mathbb{N}\}$ of natural numbers, where $(q_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence. Moreover if $b = \{k_1 < \dots < k_n < \dots\}$ is a branch of this tree, then the node $\{k_1 < \dots < k_n\}$ is symbolized as $b|n$.

Let $k \in \mathbb{N}$ and $s \in [\mathbb{N}]^{<\omega}$ with $s = \{m_1 < \dots < m_k\}$. If $m \in \mathbb{N}$ with $m_k < m$, then the finite subset $\{m_1 < \dots < m_k < m\}$, is symbolized as $(s \smallfrown m)$.

Definition 5.1. Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. A family of vectors $(x_s)_{s \in [\mathbb{N}]^{<\omega}}$ in X is called normalized weakly null tree family in X if:

- i. $\|x_s\| = 1$ for all $s \in [\mathbb{N}]^{<\omega}$ and
- ii. for each node $s \in [\mathbb{N}]^{<\omega}$ the sequence $(x_{(s \smallfrown n)})_{n \in \mathbb{N}}$ is weakly null.

Definition 5.2. Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space. We say that X has the c_0 tree property if there exists $K > 0$ such that for every normalized weakly null tree family $(x_s)_{s \in [\mathbb{N}]^{<\omega}}$ in X there exist a branch b of the tree $([\mathbb{N}]^{<\omega}, \sqsubseteq)$ such that the sequence $(x_{b|n})_{n \in \mathbb{N}}$ is K -equivalent to the usual basis of $c_0(\mathbb{N})$.

We pass to state Kalton's Theorem ([K], Thm. 3.2).

Theorem 5.1. Let X be a separable Banach space not containing $\ell^1(\mathbb{N})$. If X has the c_0 tree property, then X is embedded into c_0 .

Remark 5.1. Kalton's theorem provides an efficient characterization of the subspaces of c_0 . Similar results for ℓ^p spaces have been proved by Odell and Schlumprecht [OS2] and for subspaces of reflexive spaces with an unconditional basis by W. B. Johnson and B. Zheng [JZ].

Proposition 5.3. Let Y be a subspace of \mathfrak{X}_0 such that the identity operator $id|_Y : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ is compact. Then Y has the c_0 tree property, hence Y is embedded isomorphically into c_0 .

Proof: Let $\epsilon > 0$ and $(x_a)_{a \in [\mathbb{N}]^{<\omega}}$ be a normalized weakly null tree family in $(Y, \|\cdot\|_{W_0})$. We will construct by induction two strictly increasing sequences $(l_k)_{k \in \mathbb{N}}$ and $(j_k)_{k \in \mathbb{N}}$ of natural numbers and a block sequence $(z_k)_{k \in \mathbb{N}}$ such that:

- (i) $\frac{1}{2} \leq \|z_k\|_{W_0} \leq 2$ for all $k \in \mathbb{N}$
- (ii) $\frac{|\text{supp } z_k|}{m_{j_k}} < \epsilon_k$, for all $k \in \mathbb{N}$
- (iii) $\|z_k\|_{G_0} \leq \frac{\epsilon_{k-1}}{n_{j_{k-1}}}$ for all $k \in \mathbb{N}$ with $k \geq 2$
- (iv) setting $s_k = \{l_1 < \dots < l_k\}$ for all $k \in \mathbb{N}$, then $\|x_{s_k} - z_k\|_{W_0} < \epsilon_k$ for all $k \in \mathbb{N}$,

where $(\epsilon_k)_{k \in \mathbb{N}}$ is a sequence of positive numbers with $\epsilon_k < \frac{1}{2}$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \epsilon_k < \frac{\epsilon}{8}$.

Indeed, let $l_1 \in \mathbb{N}$. Since $x_{l_1} \in \mathfrak{X}_0$ there exists a block vector z_1 such that $\|x_{l_1} - z_1\|_{W_0} < \epsilon_1$. We choose $j_1 \in \mathbb{N}$ such that $\frac{|\text{supp } z_1|}{m_{j_1}} < \epsilon_1$. It is obvious that condition (i) is satisfied.

We assume that for a fixed $k \in \mathbb{N}$ with $k \geq 2$ we have constructed a finite increasing sequence $(l_i)_{i=1}^k$ of natural numbers, a finite sequence $(j_i)_{i=1}^k$ of increasing natural numbers and a finite block sequence $(z_i)_{i=1}^k$ such that are satisfied conditions (i)-(iv) until the fixed $k \in \mathbb{N}$.

The sequence $(x_{(s_k \widehat{\ } n)})_{n \in \mathbb{N}}$ is weakly null and normalized, therefore there exists a subsequence $(x_{(s_k \widehat{\ } q_n)})_{n \in \mathbb{N}}$ of $(x_{(s_k \widehat{\ } n)})_{n \in \mathbb{N}}$ and a block sequence $(z_n^k)_{n \in \mathbb{N}}$ such that $\|x_{(s_k \widehat{\ } q_n)} - z_n^k\|_{W_0} < r_n$ for all $n \in \mathbb{N}$, where $(r_n)_{n \in \mathbb{N}}$ is a null sequence of positive numbers.

Since the sequence $(x_{(s_k \widehat{\ } q_n)})_{n \in \mathbb{N}}$ is weakly null and the identity operator $id : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ is compact, it follows that $\lim_n \|z_n^k\|_{G_0} = 0$.

Therefore there exists $n_k \in \mathbb{N}$ with $q_{n_k} > l_k$ such that $\|z_{n_k}^k\|_{G_0} < \frac{\epsilon_k}{n_{j_k}}$, $z_k < z_{n_k}^k$ and $\|x_{(s_k \widehat{\ } q_{n_k})} - z_{n_k}^k\|_{W_0} < \epsilon_{k+1}$.

We set $z_{k+1} = z_{n_k}^k$ and $l_{k+1} = q_{n_k}$. We choose $j_{k+1} \in \mathbb{N}$ with $j_k < j_{k+1}$ such that $\frac{|\text{supp } z_{k+1}|}{m_{j_{k+1}}} < \epsilon_{k+1}$.

Thus we have constructed a block sequence $(z_k)_{k \in \mathbb{N}}$ and a branch $\{l_1 < \dots < l_k < \dots\}$ of the tree $[\mathbb{N}]^{<\omega}$, such that the sequence $(x_{s_k})_{k \in \mathbb{N}}$ is 4-equivalent to $(z_k)_{k \in \mathbb{N}}$. Conditions (i), (ii), (iii) yield that the block sequence $(z_k)_{k \in \mathbb{N}}$ is $4 + \frac{\epsilon}{4}$ equivalent to the usual basis $(e_k)_{k \in \mathbb{N}}$ of c_0 (Lemma 5.1) and therefore $(x_{s_k})_{k \in \mathbb{N}}$ is $16 + \epsilon$ equivalent to the usual basis $(e_k)_{k \in \mathbb{N}}$ of c_0 .

Also, since Y is separable, does not contain an isomorphic copy of $\ell^1(\mathbb{N})$ and has the c_0 tree property, it follows from Thm. 5.1 that Y is embedded isomorphically into c_0 .

Proposition 5.4. Let Y, Z be subspaces of \mathfrak{X}_0 such that the identity operators $id|_Y : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$, $id|_Z : (Z, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ are not compact. Then $d(S_Y, S_Z) = 0$.

Proof: We shall show that for every $\epsilon > 0$ there exist $y \in Y, z \in Z$ with $\|z - y\|_{W_0} < \epsilon \cdot \|z + y\|_{W_0}$. Then as is known this yields the result.

From the fact that $id|_Y : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ is not compact and Y does not contain an isomorphic copy of $\ell^1(\mathbb{N})$, it follows that there exist a $\|\cdot\|_{W_0}$ normalized sequence $(y_n)_{n \in \mathbb{N}}$ in Y and a block sequence $(x_n)_{n \in \mathbb{N}}$ in \mathfrak{X}_0 such that these sequences are equivalent and the identity operator $id|_{(\overline{\text{span}}\{x_n : n \in \mathbb{N}\}, \|\cdot\|_{W_0})}$ is not compact. The same holds for the subspace Z . Hence using the well known gliding hump argument, we may assume that Y, Z are block subspaces.

Since the identity operators are not compact, it follows that there exist block sequences $(y_n)_{n \in \mathbb{N}}$ in Y , $(z_n)_{n \in \mathbb{N}}$ in Z and $\epsilon_0 > 0$ such that

$$\epsilon_0 < \|y_n\|_{G_0} \leq \|y_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}$$

and

$$\epsilon_0 < \|z_n\|_{G_0} \leq \|z_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Therefore, Proposition 4.4 yields that for every $j \in \mathbb{N}$ there exists a pair (w, f) with $w \in \langle y_n : n \in \mathbb{N} \rangle$ or $w \in \langle z_n : n \in \mathbb{N} \rangle$ which is $(15, 2j, \frac{1}{2})$ exact pair in \mathfrak{X}_0 .

Hence if $j_0 \in \mathbb{N}$, then there exists a finite sequence of pairs $(w_k, f_k)_{k=1}^{n_{2j_0+1}}$ which is $(15, 2j_0 + 1, \frac{1}{2})$ dependent sequence in \mathfrak{X}_0 with $\text{ran}(f_k) = \text{ran}(w_k)$ for all $k = 1, \dots, n_{2j_0+1}$ such that for every $k \in \{1, \dots, n_{2j_0+1}\}$ we have that $w_k \in \langle y_n : n \in \mathbb{N} \rangle$ if k is odd and $w_k \in \langle z_n : n \in \mathbb{N} \rangle$ if k is even (Prop. 4.5).

From Proposition 4.6 we get that

$$(5.2) \quad \left\| \frac{1}{n_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} (-1)^i w_i \right\|_{W_0} \leq \frac{120}{m_{2j_0+1}^2}.$$

Also we have that

$$(5.3) \quad \left\| \frac{1}{n_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} w_i \right\|_{W_0} \geq \left(\frac{1}{m_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} f_i \right) \left(\frac{1}{n_{2j_0+1}} \sum_{i=1}^{n_{2j_0+1}} w_i \right) \geq \frac{1}{2m_{2j_0+1}}.$$

We set

$$A = \{i \in \{1, \dots, n_{2j_0+1}\} : i \text{ even}\} \text{ and } B = \{i \in \{1, \dots, n_{2j_0+1}\} : i \text{ odd}\}.$$

Moreover setting $y = \frac{1}{n_{2j_0+1}} \sum_{i \in B} w_i$ and $z = \frac{1}{n_{2j_0+1}} \sum_{i \in A} w_i$, from (6.13) and (5.3), we get that

$$\|z - y\|_{W_0} < \frac{240}{m_{2j_0+1}} \cdot \|z + y\|_{W_0}$$

which implies that $\|z - y\|_{W_0} < \epsilon \cdot \|z + y\|_{W_0}$ for fixed $\epsilon > 0$ and a sufficiently large $j_0 \in \mathbb{N}$.

Corollary 5.3. *Let Y, Z be subspaces of \mathfrak{X}_0 , such that the direct sum $Y \oplus Z$ is a closed subspace of \mathfrak{X}_0 . Then at least one of the identity operators $\text{id}|_Y : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$, $\text{id}|_Z : (Z, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ is compact.*

Hence one at least of the subspaces is embedded isomorphically into c_0 .

Proof: Assume on the contrary. Then from Proposition 5.4 we get that $d(S_Y, S_Z) = 0$, a contradiction. Moreover from Proposition 5.3 we obtain that one at least of the subspaces is embedded isomorphically into c_0 .

Remark 5.2. *Let Y, Z be subspaces of \mathfrak{X}_0 such that $\mathfrak{X}_i = Y \oplus Z$. Then from the fact that the identity operator $\text{id} : \mathfrak{X}_0 \rightarrow \mathfrak{X}_{G_0}$ is not compact, it follows that one at least of the identity operators $\text{id}|_Y : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ or $\text{id}|_Z : (Z, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ is not compact. Hence Corollary 5.3 yields that $\text{id}|_Y$ is compact and $\text{id}|_Z$ is not compact or vice versa.*

Corollary 5.4. *Let Y be a subspace of \mathfrak{X}_0 such that the identity operator $\text{id}|_Y : (Y, \|\cdot\|_{W_0}) \rightarrow \mathfrak{X}_{G_0}$ is not compact. Then $(Y, \|\cdot\|_{W_0})$ is not embedded isomorphically into c_0 .*

Proof: Suppose not. Then $(Y, \|\cdot\|_{W_0})$ embeds isomorphically into $c_0(\mathbb{N})$. Since the operator $\text{id}|_Y$ is not compact and Y does not contain an isomorphic copy of $\ell^1(\mathbb{N})$, it follows that there exist a sequence $(y_n)_{n \in \mathbb{N}}$ in Y and a block sequence $(x_n)_{n \in \mathbb{N}}$ in \mathfrak{X}_0 such that these sequences are equivalent and the identity operator $\text{id}|_{(\overline{\text{span}}\{x_n : n \in \mathbb{N}\}, \|\cdot\|_{W_0})}$ is not compact. Hence there exist a block sequence $(w_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and $\epsilon > 0$ such that

$$\epsilon < \|w_n\|_{G_0} \leq \|w_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Since the basis of \mathfrak{X}_0 is shrinking, we obtain that $(w_n)_{n \in \mathbb{N}}$ is weakly null in \mathfrak{X}_0 . From the fact that $(\overline{\text{span}}\{x_n : n \in \mathbb{N}\}, \|\cdot\|_{W_0})$ is embedded isomorphically into c_0 , we get (without loss of generality) that $(w_n)_{n \in \mathbb{N}}$ is equivalent to the usual basis of $c_0(\mathbb{N})$. Hence there exist $m, M > 0$ such that

$$(5.4) \quad m \cdot \max\{|a_i|, i = 1, \dots, n\} \leq \left\| \sum_{i=1}^n a_i w_i \right\|_{W_0} \leq M \cdot \max\{|a_i|, i = 1, \dots, n\}$$

for all $n \in \mathbb{N}$ and a_1, \dots, a_n real numbers.

For every $n \in \mathbb{N}$ there exists $f_n \in G_0$ such that $\text{ran}(f_n) \subset \text{ran}(w_n)$ and $f_n(w_n) > \epsilon$.

Thus setting

$$g_j = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i, j \in \mathbb{N}$$

we get that $g_j \in G_0$ and $\epsilon \leq \|\frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} w_i\|_{G_0}$ for all $j \in \mathbb{N}$. From (5.4) we have that

$$\|\frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} w_i\|_{G_0} \leq M \cdot \frac{m_{2j}}{n_{2j}}, \text{ for all } j \in \mathbb{N}, \text{ a contradiction for a sufficiently large } j \in \mathbb{N}.$$

We pass to the definition of type I and type II complemented subspaces of \mathfrak{X}_0 .

Definition 5.3. Let Y be a complemented subspace of \mathfrak{X}_0 . We say that

- i. Y is of type I if it contains an isomorph of \mathfrak{X}_0 .
- ii. Y is of type II if it is isomorphic to a subspace of c_0 .

Theorem 5.2. Let Y, Z be subspaces of \mathfrak{X}_0 such that $\mathfrak{X}_0 = Y \oplus Z$. Then Y is of type I and Z is of type II or vice versa. If especially $Y \cong c_0$, then $Z \cong \mathfrak{X}_0$.

Proof: From Corollary 5.3, Remark 5.2 and Corollary 5.4 it follows that Y is embedded isomorphically into c_0 and Z is not embedded isomorphically into $c_0(\mathbb{N})$ or vice versa. Assume without loss of generality the first case. Then there exists a subspace Z_1 of Z such that $Z \cong c_0 \oplus Z_1$. Hence

$$\mathfrak{X}_0 = Y \oplus Z \cong Y \oplus c_0 \oplus Z_1$$

and since Y is embedded isomorphically into c_0 and $c_0 \oplus c_0 \cong c_0$, we get the conclusion.

In the special case where $Y \cong c_0$ we have that

$$\mathfrak{X}_0 = Y \oplus Z \cong c_0 \oplus c_0 \oplus Z_1 \cong c_0 \oplus Z_1 \cong Z.$$

P. Koszmider [KO], under CH, has constructed a nonseparable $C(K)$ space satisfying the property that whenever Y, Z be subspaces of $C(K)$ such that $C(K) = Y \oplus Z$, then either $Y \cong c_0$ and $Z \cong C(K)$ or vice versa. In the same paper he asked whether a separable Banach space could occur sharing similar properties. The answer of this problem is affirmative and was given by S. A. Argyros and Th. Raikotsalis in [AR]. In this paper it is introduced a new class of primary Banach spaces called quasi-prime. An infinite dimensional Banach space X is said to be primary, if Y, Z be closed subspaces of X such that $X = Y \oplus Z$, then $Y \cong X$ or $Z \cong X$. The quasi-prime Banach spaces are spaces which satisfy a property like the above $C(K)$. In the present paper, the Banach space \mathfrak{X}_0 , resembles the quasi-prime Banach spaces as it seems in Thm. 5.2.

Corollary 5.5. The basis $(e_n)_{n \in \mathbb{N}}$ of \mathfrak{X}_0 , is a normalized, weakly null sequence without unconditional subsequence.

Proof: Assume the contrary. Then there exists an $L \in [\mathbb{N}]$ such that $(e_n)_{n \in L}$ is unconditional. Let L_1, L_2 in $[L]$ with $L_1 \cap L_2 = \emptyset$ and $L = L_1 \cup L_2$. Then

$$\overline{\langle e_n, n \in L \rangle} = \overline{\langle e_n, n \in L_1 \rangle} \oplus \overline{\langle e_n, n \in L_2 \rangle}.$$

From Corollary 5.3 we obtain that one at least of the identity operators $id|_{\overline{\langle e_n, n \in L_1 \rangle}}, id|_{\overline{\langle e_n, n \in L_2 \rangle}}$ is compact. Let $id|_{\overline{\langle e_n, n \in L_1 \rangle}}$ is compact. Since the basis of \mathfrak{X}_0 is shrinking, it follows that $(e_n)_{n \in L_1}$ is weakly null in \mathfrak{X}_0 and from the compactness of the identity operator we get that $\lim_{n \in L_1} \|e_n\|_{G_0} = 0$, a contradiction.

Remark 5.3. The space \mathfrak{X}_0 is not embedded into a space with an unconditional basis.

6. The space $\mathfrak{L}(\mathfrak{X}_0)$

In this section we study the structure of the operators of the space \mathfrak{X}_0 . Since the space \mathfrak{X}_0 is c_0 saturated it admits many projections. The aim is to show Theorem 6.1, which asserts that beyond the identity the non strictly singular operators are isomorph only on subspaces of \mathfrak{X}_0 which are embedded into c_0 .

Notation 6.1. Let $x \in \mathfrak{X}_0$ and Y be a subspace of \mathfrak{X}_0 . We note by $d_{G_0}(x, Y)$ the following number

$$d_{G_0}(x, Y) = \inf\{\|x - y\|_{G_0}, y \in Y\}.$$

Lemma 6.1. Let $(e_n)_{n \in \mathbb{N}}$ be the basis of \mathfrak{X}_{G_0} and \mathfrak{X}_0 and $T : \mathfrak{X}_0 \rightarrow \mathfrak{X}_0$ be a bounded linear operator.

Then $\lim_{n \rightarrow \infty} d_{G_0}(T(e_n), \mathbb{R}e_n) = 0$.

Proof: Assume on the contrary. Then there exist $\delta > 0$ and an infinite subset L of \mathbb{N} such that

$$(6.1) \quad 0 < \delta \leq d_{G_0}(T(e_n), \mathbb{R}e_n) = \inf\{\|T(e_n) - x\|_{G_0}, x \in \mathbb{R}e_n\} \text{ for all } n \in L.$$

Since $\delta \leq \|T(e_n)\|_{G_0}$ for all $n \in L$ and $(T(e_n))_{n \in \mathbb{N}}$ is weakly null in \mathfrak{X}_0 we may assume without loss of generality that there exist a block sequence $(x_n)_{n \in L}$ such that

$$(6.2) \quad \|T(e_n) - x_n\|_{W_0} < \epsilon_n \text{ for all } n \in L$$

where $(\epsilon_n)_{n \in L}$ is a sequence of positive numbers with $\epsilon_n < \frac{\delta}{2}$ for all $n \in L$ and $\sum_{n=1}^{\infty} \epsilon_n \leq 1$.

For every $n \in L$ we split x_n into tree vectors as follows

$$x_n = x_n^{(1)} + x_n^{(2)} + x_n^{(3)} \text{ with } x_n^{(i)} = E_n^{(i)} x_n \text{ for } i = 1, 2, 3$$

where $E_n^{(1)} = \{k \in \mathbb{N} : k < n\}$, $E_n^{(3)} = \{k \in \mathbb{N} : k > n\}$ and $E_n^{(2)} = \{n\}$.

From inequalities (6.1) and (6.2) we conclude that for every $n \in L$ either $\frac{\delta}{4} < \|x_n^{(1)}\|_{G_0}$ or $\frac{\delta}{4} < \|x_n^{(3)}\|_{G_0}$. Hence we may assume without loss of generality that $\frac{\delta}{4} < \|x_n^{(1)}\|_{G_0}$ for all $n \in L$ or $\frac{\delta}{4} < \|x_n^{(3)}\|_{G_0}$ for all $n \in L$.

Let $\frac{\delta}{4} < \|x_n^{(1)}\|_{G_0}$ for all $n \in L$. In the other case the proof is similar as follows.

Since $(x_n^{(1)})_{n \in L_0}$ is block, we get that for every $n \in L_0$ there exists $g_n \in G_0$ with $\text{ran}(g_n) \subset \text{ran}(x_n^{(1)})$ and $g_n(x_n^{(1)}) > \frac{\delta}{4}$. Moreover we observe that $g_n(x_n^{(1)}) = g_n(x_n)$ for all $n \in L_0$.

Let $j \in \mathbb{N}$ and $k_1 < \dots < k_{n_{2j}}$ natural numbers in L_0 with $\text{ran}(x_{k_1}) < k_1$ and $k_i < \text{ran}(x_{k_{i+1}}) < k_{i+1}$ for $i = 1, \dots, n_{2j} - 1$. Setting

$$y_j = \frac{m_{2j}}{n_{2j}} \sum_{r=1}^{n_{2j}} x_{k_r}, z_j = \frac{m_{2j}}{n_{2j}} \sum_{r=1}^{n_{2j}} e_{k_r} \text{ and } f_j = \frac{1}{m_{2j}} \sum_{r=1}^{n_{2j}} g_{k_r}$$

we observe that

- i. $\frac{\delta}{4} \leq f_j(y_j)$ and
- ii. $f_j(z_j) = 0$.

Also from (6.2) we have that

$$(6.3) \quad \|T(z_j) - y_j\|_{W_0} \leq \frac{m_{2j}}{n_{2j}} \sum_{r=1}^{n_{2j}} \epsilon_r \leq \frac{1}{m_{2j}}.$$

Let $j_0 \in \mathbb{N}$. Then we construct a finite sequence of pairs $(z_j, f_j)_{j=1}^{n_{2j_0}+1}$ which is a $(15, 2j_0 + 1, 0)$ dependent sequence in \mathfrak{X}_0 . Hence Proposition 4.6 yields that

$$(6.4) \quad \left\| \frac{m_{2j_0+1}}{n_{2j_0+1}} \sum_{j=1}^{n_{2j_0}+1} z_j \right\|_{W_0} \leq \frac{120}{m_{2j_0+1}}$$

Therefore from inequality (6.3) and (6.4) we get that

$$\begin{aligned} \frac{\delta}{4} &\leq \left\| \frac{m_{2j_0+1}}{n_{2j_0+1}} \sum_{j=1}^{n_{2j_0+1}} y_j \right\|_{W_0} \leq \left\| \frac{m_{2j_0+1}}{n_{2j_0+1}} \sum_{j=1}^{n_{2j_0+1}} (y_j - T(z_j)) \right\|_{W_0} + \left\| \frac{m_{2j_0+1}}{n_{2j_0+1}} \sum_{j=1}^{n_{2j_0+1}} T(z_j) \right\|_{W_0} \\ &\leq \frac{1}{m_{2j_0+1}} + \|T\| \left\| \frac{m_{2j_0+1}}{n_{2j_0+1}} \sum_{j=1}^{n_{2j_0+1}} z_j \right\|_{W_0} \leq \frac{120\|T\| + 1}{m_{2j_0+1}} \end{aligned}$$

which is a contradiction for a sufficiently large $j_0 \in \mathbb{N}$.

Lemma 6.2. *Let $T : \mathfrak{X}_0 \longrightarrow \mathfrak{X}_0$ be a bounded linear operator and $L \in [\mathbb{N}]$ such that*

- i. $\lim_{n \in L} \|T(e_n)\|_{G_0} = 0$ and
- ii. *there exists a subspace Y of \mathfrak{X}_0 such that the restriction $T|_Y : Y \longrightarrow T(Y)$ of T is an isomorphism.*

Then the identity operator $id : (Y, \|\cdot\|_{W_0}) \longrightarrow \mathfrak{X}_{G_0}$ is compact.

Proof: Suppose not. Then as in Proposition 5.4 we may assume without loss of generality that Y is a block subspace of \mathfrak{X}_0 . Hence there exists a block sequence $(y_n)_{n \in \mathbb{N}}$ in Y and $\epsilon_0 > 0$ such that

$$\epsilon_0 < \|y_n\|_{G_0} \leq \|y_n\|_{W_0} \leq 1 \text{ for all } n \in \mathbb{N}.$$

Since $T|_Y$ is an isomorphism, there exists $m > 0$ such that

$$(6.5) \quad m\|x\|_{W_0} \leq \|T(x)\|_{W_0} \leq \|T\|\|x\|_{W_0} \text{ for every } x \in Y.$$

We distinguish the following cases.

1. Let $\lim_{n \in L} \|T(e_n)\|_{W_0} = 0$.

There exists $M \in [L]$ such that

$$(6.6) \quad \sum_{n \in M} \|T(e_n)\|_{W_0} < \frac{m}{2}.$$

From the fact that the identity operators $id|_Y : (Y, \|\cdot\|_{W_0}) \longrightarrow \mathfrak{X}_{G_0}$, and $id|_{\overline{\langle e_n, n \in M \rangle}} : (\overline{\langle e_n, n \in M \rangle}, \|\cdot\|_{W_0}) \longrightarrow \mathfrak{X}_{G_0}$ are not compact, Prop. 5.4 yields that there exist $y \in Y$ with $\|y\|_{W_0} = 1$ and $z \in \langle e_n, n \in M \rangle$ with $\|z\|_{W_0} = 1$ such that $\|y - z\|_{W_0} < \frac{m}{2\|T\|}$.

Therefore

$$(6.7) \quad \|T(y - z)\|_{W_0} \leq \|T\| \cdot \|y - z\|_{W_0} < \frac{m}{2}.$$

Let $z = \sum_{i \in F} a_i e_i$, where F is a finite subset of M . Then using (6.6) we get that

$$(6.8) \quad \|T(z)\|_{W_0} \leq \sum_{i \in F} \|T(e_i)\|_{W_0} < \frac{m}{2}.$$

Hence from the isomorphism and (6.8) we have that

$$(6.9) \quad \|T(y - z)\|_{W_0} \geq \|T(y)\|_{W_0} - \|T(z)\|_{W_0} \geq m - \frac{m}{2} = \frac{m}{2}$$

which contradicts to (6.7).

2. Let $\limsup_{n \in L} \|T(e_n)\|_{W_0} > 0$.

We set $L = \{l_1 < \dots < l_n < \dots\}$. Since $(T(e_n))_{n \in \mathbb{N}}$ is weakly null in \mathfrak{X}_0 , we may assume without loss of generality that there exist a block sequence $(x_n)_{n \in L}$ which is equivalent to $(T(e_n))_{n \in L}$. Hence the block sequence $(x_n)_{n \in L}$ is $\|\cdot\|_{W_0}$ -bounded, $\|\cdot\|_{W_0}$ -seminormalized and $\lim_{n \in L} \|x_n\|_{G_0} = 0$.

From Lemma 5.2 there exists a subsequence of $(x_n)_{n \in L}$ which is equivalent to the usual basis of c_0 . We may assume without loss of generality that $(x_n)_{n \in L}$ is equivalent to the usual basis of $c_0(\mathbb{N})$. Moreover, since $(x_n)_{n \in L}$ is equivalent to $(T(e_n))_{n \in L}$, it follows that there exist $d_1, d_2 > 0$ such that

$$(6.10) \quad d_1 \cdot \max\{|a_i| : i = 1, \dots, n\} \leq \left\| \sum_{i=1}^n a_i T(e_i) \right\|_{W_0} \leq d_2 \cdot \max\{|a_i| : i = 1, \dots, n\}$$

for every $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{R}$.

We choose $j_0 \in \mathbb{N}$ such that $\frac{1}{m_{2j_0+1}} < \min\{\frac{m}{8d_2}, \frac{m}{8 \cdot 120 \|T\|}\}$.

Hence from Prop. 4.5 there exists a $(15, 2j_0 + 1, \frac{1}{2})$ dependent sequence $(w_k, f_k)_{k=1}^{n_{2j_0+1}}$ in \mathfrak{X}_0 with $\text{ran}(f_k) = \text{ran}(w_k)$ for all $k = 1, \dots, n_{2j_0+1}$ such that for every $k \in \{1, \dots, n_{2j_0+1}\}$ we have that $w_k \in \langle y_n : n \in \mathbb{N} \rangle$ if k is odd and $w_k \in \langle e_n : n \in \mathbb{N} \rangle$ if k is even.

Let $j_1 \in \Omega_1$ with $n_{2j_0+1}^2 < m_{2j_1}$. Then from Proposition 4.4, there exists a $(15, 2j_1, \frac{1}{2})$ exact pair (w_1, f_1) in \mathfrak{X}_0 , with $w_1 \in \langle y_n : n \in \mathbb{N} \rangle$.

Let $j_2 = \sigma(f_1)$ and $F_2 \subset L$ with $\#F_2 = n_{2j_2}$ and $F_2 > \text{maxsupp}(w_1)$.

We set $w_2 = \frac{m_{2j_2}}{n_{2j_2}} \sum_{i \in F_2} e_i$ and $f_2 = \frac{1}{m_{2j_2}} \sum_{i \in F_2} e_i^*$. Then the pair (w_2, f_2) is a $(15, 2j_2, \frac{1}{2})$ exact pair in \mathfrak{X}_0 .

In this way we inductively construct the $(15, 2j_0 + 1, \frac{1}{2})$ dependent sequence $(w_k, f_k)_{k=1}^{n_{2j_0+1}}$ in \mathfrak{X}_0 .

From Proposition 4.6 we get that

$$\left\| \frac{1}{n_{2j_0+1}} \sum_{k=1}^{n_{2j_0+1}} (-1)^k w_k \right\|_{W_0} \leq \frac{120}{m_{2j_0+1}^2}.$$

Hence

$$(6.11) \quad \left\| T \left(\frac{1}{n_{2j_0+1}} \sum_{k=1}^{n_{2j_0+1}} (-1)^k w_k \right) \right\|_{W_0} \leq \frac{120 \|T\|}{m_{2j_0+1}^2}.$$

We set

$$A = \{k \in \{1, \dots, n_{2j_0+1}\} : k \text{ even}\} \text{ and } B = \{k \in \{1, \dots, n_{2j_0+1}\} : k \text{ odd}\}.$$

Then

$$(6.12) \quad \left\| \frac{1}{n_{2j_0+1}} \cdot \sum_{k \in B} w_k \right\|_{W_0} \geq \left(\frac{1}{m_{2j_0+1}} \sum_{k=1}^{n_{2j_0+1}} f_k \right) \left(\frac{1}{n_{2j_0+1}} \cdot \sum_{k \in B} w_k \right) \geq \frac{1}{4m_{2j_0+1}}.$$

and inequality (6.10) yields that

$$(6.13) \quad \|T(w_k)\|_{W_0} \leq \frac{m_{2j_k}}{n_{2j_k}} d_2 \leq \frac{1}{m_{2j_0+1}^2} d_2 \leq \frac{m}{8m_{2j_0+1}} \text{ for every } k \in A.$$

Therefore from (6.5), (6.11), (6.12), (6.13) and the triangle inequality we get that

$$\frac{120 \|T\|}{m_{2j_0+1}^2} \geq m \left\| \frac{1}{n_{2j_0+1}} \cdot \sum_{k \in B} w_k \right\|_{W_0} - \left\| T \left(\frac{1}{n_{2j_0+1}} \cdot \sum_{k \in A} w_k \right) \right\|_{W_0} \geq \frac{m}{4m_{2j_0+1}} - \frac{m}{8m_{2j_0+1}} = \frac{m}{8m_{2j_0+1}}$$

which contradicts to the choice of $j_0 \in \mathbb{N}$.

Theorem 6.1. *Let $T : \mathfrak{X}_0 \longrightarrow \mathfrak{X}_0$ be a bounded linear operator. Then $T = \lambda I + S$, where $\lambda \in \mathbb{R}$, I the identity operator on \mathfrak{X}_0 and S be a bounded linear operator on \mathfrak{X}_0 such that whenever Y is a subspace of \mathfrak{X}_0 with $S|_Y : Y \longrightarrow S(Y)$ is an isomorphism it follows that the subspace Y is embedded isomorphically into $c_0(\mathbb{N})$.*

Proof: From Lemma 6.1 it follows that there exist an $L \in [\mathbb{N}]$ and a sequence $(\lambda_n)_{n \in L}$ of real numbers such that $\lim_{n \in L} \|T(e_n) - \lambda_n e_n\|_{G_0} = 0$. We observe that the sequence $(\lambda_n)_{n \in L}$ is bounded and consequently there exists $M \in [L]$ and $\lambda \in \mathbb{R}$ such that $\lim_{n \in M} \lambda_n = \lambda$. It is not hard to see that

$\lim_{n \in M} \|T(e_n) - \lambda e_n\|_{G_0} = 0$. Setting $S = T - \lambda I$, if S restricted in some subspace is an isomorphism, then from Lemma 6.2 and Proposition 5.3 it follows the conclusion.

Remark 6.1. *In a forthcoming paper we will present some variants of the space \mathfrak{X}_0 . More precisely for $1 \leq p < \infty$ we construct an ℓ^p saturated Banach space \mathfrak{X}_p such that for every $p > 1$ the space \mathfrak{X}_p is reflexive and satisfies tightness conditions similar to the corresponding ones of \mathfrak{X}_0 .*

REFERENCES

- [AKT] S. A. Argyros, B. Kanellopoulos, K. Tyros, *Higher order spreading models in Banach space theory*, (to appear), arXiv:1006.0957.
- [AMP] S. A. Argyros, A. Manoussakis, A. M. Pelczar, *On the hereditary proximity to ℓ_1* , (to appear), arXiv:0907.4317.
- [ATO] S. A. Argyros, S. Todorćević, *Ramsey methods in analysis*, Birkhauser, CRM, Barcelona, 2004.
- [AR] S. A. Argyros, Th. Raikoftsalis, *Banach spaces with a unique nontrivial decomposition*, Proc. AMS, Vol. 136 (2008), no. 10, 3611-3620.
- [GM] W. T. Gowers, B. Maurey, *The unconditional basic sequence problem*, Jour. A.M.S. Vol. 6 (1993), no. 4, 851-874.
- [J] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math., Vol. 52 (1950), 518-527.
- [JZ] W. B. Johnson, B. Zheng, *A characterization of subspaces and quotients of reflexive Banach spaces with unconditional bases*, Duke Math. J., Vol. 141 (2008), 505-518.
- [K] N. J. Kalton, *On subspaces of c_0 and extensions of operators into $C(K)$ -spaces*, Q. J. Math., Vol. 52 (2001), 312-328.
- [KO] P. Koszmider, *On decompositions of Banach spaces of continuous functions on Mrowka's spaces*, Proc. AMS, Vol. 133 (2005), 2137-2146.
- [OS1] E. Odell, Th. Schlumprecht, *On the richness of the set of p 's in Krivine's theorem*, Geometric aspects of functional analysis (Israel, 1992-1994), 177-198, Oper. Theory Adv. Appl., 77, Birkhauser, Basel, 1995.
- [OS2] E. Odell, Th. Schlumprecht, *Tree and branches in Banach spaces*, Trans. AMS, Vol. 354 (2002), no. 10, 4085-4108.
- [R] F. P. Ramsey, *On a problem of formal logic*, Proc. LMS, Vol. 30 (1930), 264-286.
- [SO] A. Sobczyk, *Projection of the space m on its subspace c_0* , Bull. Amer. Math. Soc., Vol. 47 (1941), 938-947.

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